# An Analytical Model for Traffic Delays

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## 1 Introduction

The dynamic traffic assignment problem relies on an accurate model for evaluating traffic delays. This paper derives an analytical function of the travel time, based on the theory of kinematic waves, starting with a single stretch of road and subsequently generalizing over a network. The travel time function integrates traffic dynamics and the effects of shocks. Numerical examples illustrate the quality of the analytical travel time model in comparison with simulated ones from the literature.

Dynamic traffic assignment models either simulate flow propagation (see Newell 1993, Daganzo 1994, Mahut 2000 and Khoo et al. 2002) or rely on an analytical travel time function (e.g., see Ran and Boyce 1994). Using an analytical travel time function allows studying the convergence behavior of algorithms.

A lot of research has been devoted to identifying the variables that affect travel time (see Ran and Boyce 1994, Daganzo 1995a and Carey et al. 2003). However, very few models propose a functional form of the travel time. Based on the kinematic wave model of Lighthill and Whitham (1955) and Richards (1956), Kachani and Perakis (2001) proposed a polynomial and an exponential travel time functions when there is no congestion. Using the simplified model of Newell (1993), Kuwahara and Akamatsu (2001) derived an analytical function for the instantaneous travel time integrating congestion effects.

In this paper, we extend the work by Kuwahara and Akamatsu (2001) and derive an analytical function of the experienced, instead of instantaneous, travel time. We also extend the work by Kachani and Perakis (2001) by integrating congestion effects into the travel time function. The main contributions of our model are the following:

- the methodology applies to both triangular and quadratic fundamental diagrams;
- the travel time function integrates first-order traffic dynamics, shocks and queue spillovers;
- from our numerical experiments, the proposed analytical travel time function behaves similarly to the travel times obtained by simulation.

In Perakis and Roels (2004), we extend our results from a single stretch of road to a general network. Furthermore, we incorporate the travel time model into a Dynamic User Equilibrium setting in order to determine the equilibrium flows.

## 2 Review of the Theory of Kinematic Waves

In this section, we review the hydrodynamic theory of traffic flow, proposed by Lighthill and Whitham (1955) and Richards (1956). Because of the dynamic nature of traffic, we work on a time-space setting. The time origin is set to $t_0$ and the road has length $L$. The fundamental traffic variables to describe traffic conditions on a road are (1) the flow rate, $f(x,t)$, which is the number of vehicles per hour passing location $x$ at time $t$, (2) the rate of density, $k(x,t)$, which is the number of vehicles per mile, at location $x$ at time $t$, and (3) the instantaneous velocity, $u(x,t)$, which is the speed of vehicles passing location $x$ at time $t$. In what follows, we assume that the road is characterized by the maximum speed $u^{max}$, the maximum density $k^{max}$, and the capacity $f^{max}$ on that road.

Notice that the definition of the fundamental traffic variables implies that $f(x,t) = k(x,t)u(x,t)$. Most models assume a one-to-one relation between the speed and the density, i.e., $u(x,t) = u(k(x,t))$. Therefore, the flow and the density are related through the so-called *fundamental diagram*:

$$f(x,t) = k(x,t)u(k(x,t)) \; \forall x, t. \tag{1}$$

Depending on the assumed speed-density relationship $u(k)$, the fundamental diagram can have different shapes. In the following, we analyze the two most common shapes, namely the quadratic and the triangular fundamental diagrams.

If the vehicle velocity depends linearly on density, i.e. $u(k) = u^{max}(1-k/k^{max})$, as in Richards (1956), the fundamental diagram has a quadratic shape:

$$f(x,t) = k(x,t)u^{max}(1 - k(x,t)/k^{max}). \tag{2}$$

On the other hand, Newell (1993) proposed a triangular curve with a left slope of $u_0$ and a right slope of $-w_0$. The change of slope occurs when the flow is at capacity.

$$f(x,t) = \begin{cases} k(x,t)/u_0 & \text{if } k(x,t) \le f^{max}u_0, \\ (k^{max} - k(x,t))/w_0 & \text{otherwise.} \end{cases} \tag{3}$$

In addition to (1), the traffic variables are related through a conservation law, stated as the following partial differential equation:

$$\frac{\partial k(x,t)}{\partial t} + \frac{df(k)}{dk}\frac{\partial k(x,t)}{\partial x} = 0. \tag{4}$$

This relation describes the fact that, on a single stretch of road, no cars are lost.

The standard way of solving (4) is the method of characteristics. Along a characteristic line, the density (and therefore the flow) remains constant. The slope of a characteristic line, $df/dk$, is positive when the traffic is light, and negative when the traffic is heavy. According to the theory of kinematic waves, a characteristic line is the trajectory of a wave, propagating forwards or backwards, and conveying constant flow. If two characteristic lines intersect, the density around the point of intersection is discontinuous. The set of such points of intersection is called a shock wave. For instance, the sudden change in traffic conditions around the shock wave may be due to an accident or a downstream bottleneck capacity.

# 3 An analytical derivation of the travel time function

## 3.1 Assumptions

Because of the discontinuity induced by shocks, the kinematic wave model may be quite hard to solve. In what follows, we introduce three assumptions that simplify the model.

**Assumption A1 - At most one shock** We assume that there is at most one shock on the road. As in Newell (1993), we assume that a shock can only result from the focusing of one forward and one backward wave. Accordingly, we can divide a road into two segments, separated by the shock wave: on the first segment, the traffic flow has a low density, whereas on the second, it has a high density. Note that the length of each segment depends on traffic conditions and is therefore time-varying.

**Assumption A2 - Linear density** We assume that the second-order variation of density is locally negligible (which is always true for small road lengths). Accordingly, the density at location $x$ at time $t$ can be approximated with $k(x,t) = k(\xi,t) + B(\xi,t)(x-\xi)$, where $\xi = 0$ if the traffic conditions at $(x,t)$ are light, and $\xi = L$ if they are heavy.

**Assumption A3 - Smooth dynamic effects** In addition, we require the dynamic effects to be smooth. If the evolution of traffic flow is highly variable, we can relax this assumption by considering smaller road lengths. Mathematically, we impose the condition that $|B(\xi,t)| < (k^{max} - k(\xi,t))^2/(5Lk^{max})$, for $\xi = 0$ or $L$.

## 3.2 Methodology

From assumption **A1**, the road can be decomposed into two segments, separated by the shock wave. As a result, the total travel time can be computed as the sum of (1) the travel time to go from the entrance to the shock wave (under light traffic conditions), and (2) the travel time to go from the shock wave to the road exit (under heavy traffic conditions).

In particular, we consider a vehicle that starts its trip at time $t_0$, on a road of length $L$. We denote by $\tau(x)$ its travel time to reach location $x$. Let $T$ be the travel time of the vehicle to reach the exit, if there was no shock; however, because of the shock, its total travel time will be significantly larger.

**Shock location** Newell (1993) introduced the concept of cumulative number of vehicles passing through location $x$ by time $t$. He showed that, along a characteristic line, the cumulative number of cars evolves linearly. If the characteristic line has positive slope, the cumulative number of vehicles along this characteristic line depends on the traffic conditions at the entrance and is denoted by $A(x,t)$. Symmetrically, if the characteristic line has negative slope, the cumulative number of vehicles along the characteristic line depends on the traffic conditions at the exit and is denoted by $D(x,t)$.

At the intersection of two characteristic lines, the cumulative number of cars is multivalued. Newell argued that the correct value is the minimum between $A(x,t)$ and $D(x,t)$, and that the path of intersection, $A(x,t) = D(x,t)$, is the shock. Plugging the vehicle's trajectory $(x, t_0 + \tau(x))$ into the shock wave equation, $A(x,t) = D(x,t)$, gives rise to the point $(\hat{x}, t_0 + \hat{\tau})$ at which the vehicle goes through the shock.

**Travel time function in light/heavy traffic** From the fundamental diagram (1), the instantaneous vehicle's velocity at $(x,t)$ is the ratio between the flow and the density. Accordingly, the vehicle's trajectory $(x, t_0 + \tau(x))$ evolves as follows:

$$\frac{d\tau(x)}{dx} = \frac{1}{u(x, t + \tau(x))} = \frac{k(x, t_0 + \tau(x))}{f(x, t_0 + \tau(x))}. \tag{5}$$

From the flow-density curve (2) or (3), $f(x, t_0 + \tau(x))$ can be expressed as a function of $k(x, t_0 + \tau(x))$. Therefore, the right hand side of (5) only depends on the density. Using assumption **A2**, we obtain an ordinary differential equation (ODE) to describe the evolution of the travel time. This ODE, together with the initial condition $\tau(0) = 0$, can be solved with a power series solution. Under assumption **A3**, the ratio between two successive terms in the power series is bounded above by 1, and the series converges.

For the heavy traffic region, the methodology for computing the travel time is similar, but takes $(L, t_0 + T)$ as a reference point, instead of $(0, t_0)$. In addition, the boundary condition refers to the shock location on the vehicle's trajectory, i.e. $\tau(\hat{x}) = \hat{\tau}$.

In what follows, we apply this general methodology to the triangular and the quadratic relations between the flow and the density.

### 3.3 Triangular fundamental diagram

**Theorem 1** *With a triangular fundamental diagram, under assumptions* **A2** *and* **A3**, *the travel time of a vehicle entering the road at time* $t_0$ *is*

$$\tau = \hat{x}u_0 + \frac{(k(L,t_0+T)w_0 - B(L,t_0+T)(L-\hat{x})(u_0+w_0))w_0}{(k^{max} - k(L,t_0+T))w_0 + B(L,t_0+T)(w_0+u_0)(L-\hat{x})}(L-\hat{x}) + O\left(\frac{Lw_0 k(L,t_0+T)}{50(k^{max} - k(L,t_0+T))}\right), \tag{6}$$

*where* $\hat{x} = \hat{x} = L - \frac{A(0,t_0) - D(L,t_0+T)}{k(L,t_0+T) - (k^{max} - k(L,t_0+T))\frac{u_0}{w_0}}$.

In Figure 1, the analytical travel time (6) behaves similarly to the travel time obtained by simulation using the Cell Transmission Model (CTM) by Daganzo (1994). In this example, we considered a quadratic entering flow rate, $f(0,t) = 1600 - 6400(t/3600 - 0.5)^2$ vehicles/hour for $t = 1, .., 3600$ seconds. The fundamental diagram is symmetric triangular, with $u_0 = w_0 = 1/u^{max} = 1/40$ hour/mile, $k^{max} = 200$ vehicles/mile, $f^{max} = 4000$ vehicles/hour. The road has a length of 4 miles and has a bottleneck at its end authorizing only 1400 vehicles/hour to exit the road.

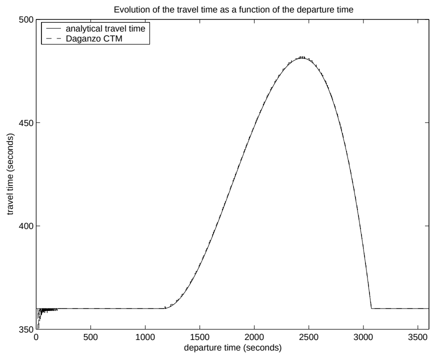


Figure 1: Evolution of the travel time with the departure time, when the entering flow is quadratic. Comparison of (6) with the CTM for a triangular fundamental diagram

### 3.4 Quadratic fundamental diagram

**Theorem 2** *With a quadratic fundamental diagram, if there is no shock, and under assumptions* **A2** *and* **A3***, the travel time of a vehicle entering a road of length $x$ at time $t_0$ is given by*

$$\tau(x) = \frac{x}{u^{max}(1 - k(0,t)/k^{max})} + \frac{1}{2}\frac{B(0,t)k(0,t)x^2}{(k^{max})^2 u^{max}(1 - k(0,t)/k^{max})^3} + O(\frac{L}{50u^{max}}). \qquad (7)$$

**Theorem 3** *With a quadratic fundamental diagram, under assumptions* **A2** *and* **A3***, the total travel time of a vehicle entering the road at time $t_0$ is*

$$\tau = \hat{\tau} + \frac{k^{max} + 2u^{max}(T - \hat{\tau})B(L, t_0 + T)}{u^{max}(k^{max} - k(L, t_0 + T) + B(L, t_0 + T)(L - \hat{x} + u^{max}(T - \hat{\tau})))}(L - \hat{x}) + O(\frac{Lk(L, t_0 + T)}{50u^{max}(k^{max} - k(L, t_0 + T))}), \qquad (8)$$

*where $\hat{x} = L - (A(0, t_0) - D(L, t_0 + T))\frac{k^{max} - k(0,t_0)}{k(L,t_0+T)(k(L,t_0+T) - k(0,t_0))}$ is the shock location, $T = \tau(L)$, $\hat{\tau} = \tau(\hat{x})$ according to (7).*

In Figure 2, we compared the analytical travel time (8) with the travel times obtained by simulation, with or without congestion (right and left figures, respectively). We considered the same quadratic entering flow rate as in the triangular case. The road has the same characteristics as in the triangular example, but has a capacity of $f^{max} = 2000$ cars/hour.

When there is no bottleneck at the exit of the road, there is no congestion, and the analytical travel time function (8) behaves similarly to the travel times simulated using the procedures described in Khoo et al. (2002) and Daganzo (1995b). In contrast, the analytical travel time proposed by Kachani and Perakis (2001) tends to underestimate the simulated travel times.

With the introduction of a bottleneck at the road exit of the road of 1500 cars/hour, the travel time is affected by congestion. As shown in the right figure, the analytical travel time (8) depicts the same behavior as the travel time simulated with Finite Difference Equation model of Daganzo. A comparison of the left and right figures illustrates how congestion can affect travel time.

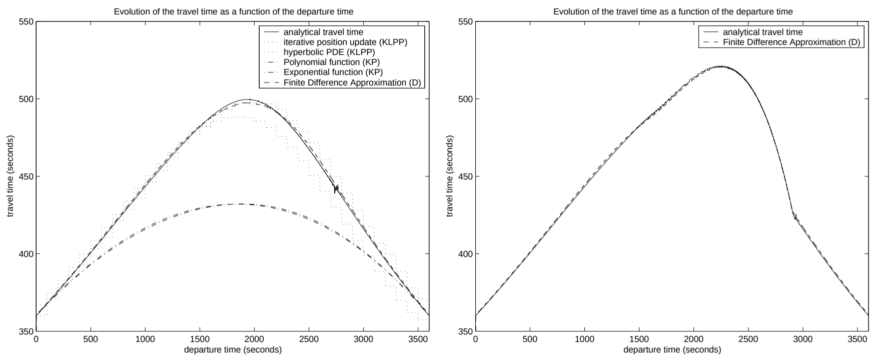


Figure 2: Evolution of the travel time with the departure time, when the entering flow is quadratic. Comparison of (8) with simulated travel times for a quadratic fundamental diagram

In Perakis and Roels (2004), we extend this methodology to compute the experienced travel time of a vehicle on a general network, and incorporate it into a Dynamic User-Equilibrium setting.

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