A Precise Sensitivity Analysis of Traffic Equilibria, with Applications

Michael Patriksson*

*Department of Mathematics, Chalmers University of Technology
SE-412 96 Gothenburg
Sweden
mipat@math.chalmers.se

1 Introduction

While the most popular sensitivity analysis tools for traffic equilibrium (TE) models are technically accessible and have managed to bring the field to practitioners, the tools are often inapplicable because of their strong requirements, and they are also impractical for use with large-scale networks. In the case of the formula constructed in [ToF88], there are examples where it does not produce any results and yet the equilibrium link flow is differentiable. In the case of the analysis presented in [CSF00], the sensitivity analysis also only works in restrictive circumstances and the computational formula is complicated. The in our opinion *natural* sensitivity analysis tool retains the structure of the original TE model, and the tool is therefore easy to construct by suitably (and mildly) adjusting an efficient TE solver: the computational formula for a directional derivative (and/or, partial gradient) is based on a linearization of the original model around the fixed perturbation, and is an affine traffic equilibrium model over a restriction of the original traffic network.

The paper has the following aims: (1) to provide simple (and not at all contrived) examples that we hope will convince the users of traffic equilibrium models to utilize the more widely applicable formulas presented here; (2) to show exactly when, and how, the correct formula can be applied, both for deterministic/logit-based, separable/non-separable and fixed/elastic demand models, and not only for equilibrium link flows but for travel costs and demands as well; (3) to apply it to a classic network design problem, thereby reaching interesting conclusions on the use of descent algorithms for bilevel programming applications in transport. The paper summarizes the published papers [PaR03, Pat04] and some forthcoming work ([JoP03, Pat05]).

2 Sensitivity analysis of variational problems

Consider the variational inequality (VI) problem

$$-f(\rho, x) \in N_C(x),$$

where $\rho \in \Re^d$ is the parameter, $x \in \Re^n$ is the solution, $f : \Re^d \times \Re^n \mapsto \Re^n$ is a smooth function, $C \subseteq \Re^n$ is a nonempty polyhedral set, and N_C is the normal cone mapping to C, that is,

$$N_C(x) = \begin{cases} \{ z \in \Re^n \mid z^{\mathrm{T}}(x - y) \leq 0, \ \forall y \in C \}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

This is a VI parameterized by ρ. We focus our attention on the generalized differentiation of the solution mapping

$$S : \rho \mapsto S(\rho) := \{ x \mid -f(\rho, x) \in N_C(x) \}$$

at a pair (ρ^*, x^*) with $x^* \in S(\rho^*)$.

The parameterization is presumed to be done in such a way that

$$\operatorname{rank} \nabla_\rho f(\rho^*, x^*) = n, \qquad \nabla_\rho f(\rho^*, x^*) \in \Re^{n \times d}.$$

(This property can always be enforced.) The *proto-derivative* of S at ρ^* for x^* in the direction of ρ' is the solution to the auxiliary VI

$$DS(\rho^* \mid x^*)(\rho') := \left\{ x' \mid r(\rho', x') + N_K(x') \ni 0^n \right\}, \quad \text{where}$$

$$r(\rho', x') := \nabla_\rho f(\rho^*, x^*)\rho' + \nabla_x f(\rho^*, x^*)x' \quad \text{and} \quad K := T_C(x^*) \cap f(\rho^*, x^*)^\perp.$$

(For a vector $z \in \Re^n$, $z^\perp := \{ y \in \Re^n \mid z^{\mathrm{T}} y = 0 \}$.) The set K in the above definition is the *critical cone* associated with the variational inequality (2) for $\rho = \rho^*$ and $x = x^*$. The above VI is a local linearization of the original problem. The main theoretical results for the directional differentiability of the solution x are given in [DoR01]. An equivalence result holds, which states that S is single-valued and Lipschitz continuous on some neighbourhood of ρ^* if and only if $DS(\rho^* \mid x^*)$ is single-valued on some neighbourhood of 0^d (hence everywhere). Moreover, then S is *semi-differentiable* at ρ^* for x^*, and $DS(\rho^* \mid x^*)$ is not only Lipschitz continuous and positively homogeneous but also piecewise linear. Moreover, the semi-differential (directional derivative) mapping DS is linear if and only if S is differentiable, in which case $DS(\rho^* \mid x^*)(\rho') = \nabla S(\rho^*)^{\mathrm{T}} \rho'$. Hence, the gradient of S at ρ^* then is obtained as the d coordinate-directional derivatives $DS(\rho^* \mid x^*)(e_i)$, where e_i $(i = 1, \ldots, d)$ is the ith unit vector.

Finally, differentiability is equivalent to $DS(\rho^* \mid x^*)(\rho') \in -K$ for every $\rho \in \Re^d$ ([Kyp90]). We apply and refine these results below.

3 Traffic (Wardrop) equilibrium

In order to apply the above theory, we formulate the traffic equilibrium model following the notation of [Pat05]. $\mathcal{G} = (\mathcal{N}, \mathcal{L})$ is the network, where $\mathcal{N}$ and $\mathcal{L}$ are the sets of nodes and directed links. $\mathcal{C}$ is a subset of $\mathcal{N} \times \mathcal{N}$, denoting the set of OD pairs (p, q). For each such pair, we have a demand function $g_{pq}(\rho, \cdot) : \Re^{|\mathcal{C}|} \mapsto \Re_+$, which is nonnegative and upper bounded and its negative strictly monotone (hence invertible). The routes in OD pair (p, q) are denoted by $\mathcal{R}_{pq}$, and their flow by h_{pqr}. The cost of travel on route r is $c_{pqr}(\rho, h)$. With

$$H_d := \left\{ (h, d) \in \Re_+^{|\mathcal{R}|} \times \Re^{|\mathcal{C}|} \;\middle|\; \Gamma^{\mathrm{T}} h = d \right\},$$

Γ being the route–OD pair incidence matrix, the Wardrop equilibrium condition is

$$[-c(\rho, h), g^{-1}(\rho, d)] \in N_{H_d}(h, d).$$

Assuming that the route cost is additive, we further have that with

$$\widehat{F}_d := \left\{ (v, d) \in \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|} \;\middle|\; \exists (h, d) \in H_d \text{ with } v = \Lambda h \right\}$$

(Λ being the route–link incidence matrix), (3) can be equivalently written as

$$[-t(\rho, v), g^{-1}(\rho, d)] \in N_{\widehat{F}_d}(v, d),$$

where $t(\rho, \cdot) : \Re_+^{|\mathcal{L}|} \mapsto \Re_{++}^{|\mathcal{L}|}$ is the link cost vector.

4 Sensitivity analysis of traffic equilibria

4.1 Formulations

We first identify the sensitivity problem in our notation. Let

$$x = \begin{pmatrix} h \\ v \\ d \end{pmatrix}; \qquad f(\rho, x) = \begin{pmatrix} 0^{|\mathcal{R}|} \\ t(\rho, v) \\ -\xi(\rho, d) \end{pmatrix}; \qquad C = \Re_+^{|\mathcal{R}|} \times \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|}.$$

Then, we can identify the sensitivity problem through the following identifications:

$$K = \left\{ \begin{pmatrix} h' \\ v' \\ d' \end{pmatrix} \in \Re^{|\mathcal{R}|} \times \Re^{|\mathcal{L}|} \times \Re^{|\mathcal{C}|} \;\middle|\; \Gamma^{\mathrm{T}} h' = d'; \quad v' = \Lambda h'; \quad h' \in H' \right\},$$

where

$$H' = \left\{ h' \in \Re^{|\mathcal{R}|} \;\middle|\; \begin{array}{l} h'_r \text{ free if } h^*_r > 0 \\ h'_r \geq 0 \text{ if } h^*_r = 0 \text{ and } c_r(\rho^*, h^*) = \pi^*_k \\ h'_r = 0 \text{ if } h^*_r = 0 \text{ and } c_r(\rho^*, h^*) > \pi^*_k \\ [r \in \mathcal{R}_k, \ k \in \mathcal{C}] \end{array} \right\},$$

and

$$r(\rho', x') = \begin{pmatrix} 0^{|\mathcal{R}|} \\ \nabla_\rho t(\rho^*, v^*)\rho' + \nabla_v t(\rho^*, v^*) v' \\ -[\nabla_\rho \xi(\rho^*, d^*)\rho' + \nabla_d \xi(\rho^*, d^*) d'] \end{pmatrix}.$$

If we have strict monotonicity and separability of t and $-\xi$, then the resulting sensitivity VI can be equivalently written as the following convex quadratic optimization problem to

$$\begin{aligned} \underset{(v', d')}{\text{minimize}}\ \phi'(v', d') := [\nabla_\rho t(\rho^*, v^*)\rho']^{\mathrm{T}} v' + \frac{1}{2} \sum_{l \in \mathcal{L}} \frac{\partial t_l(\rho^*, v^*_l)}{\partial v_l} (v'_l)^2 & \\ - [\nabla_\rho \xi(\rho^*, d^*)\rho']^{\mathrm{T}} d' - \frac{1}{2} \sum_{k \in \mathcal{C}} \frac{\partial \xi_k(\rho^*, d^*_k)}{\partial d_k} (d'_k)^2, & \qquad (1a) \end{aligned}$$

$$\text{subject to } \Gamma^{\mathrm{T}} h' = d', \qquad (1b)$$

$$v' = \Lambda h', \qquad (1c)$$

$$h \in H'. \qquad (1d)$$

The sensitivity problem is closely related to the original model. Two main differences are notable: the link cost and demand functions are affine, as a result of their linearization, and the sign restrictions on h are replaced by individual restrictions on the route flow perturbations h'_r that depend on whether the route in question was used at equilibrium or not, cf. the set H'. Although the appearance of H' depends on the choice of route flow solution h^*, it is an interesting fact that the possible choices of v' in K does *not*; this is a general consequence of aggregation. In summary, the sensitivity problem can be solved using software similar to those for the original traffic equilibrium model, provided of course that route flow information can be extracted.

The more general cases of non-invertible demand functions and/or non-separable travel cost functions, as well as the link–node representation of flows, are derived similarly.

4.2 Results

Next we list sample sensitivity results. Suppose that t and $-g$ are separable, differentiable, and strictly monotone.

- The solution (v^*, d^*) to the TEP is unique.
- The link, route, and least OD travel cost sensitivities t', c', and π' are unique.
- Suppose that $\frac{\partial t_l(\rho^*, v_l^*)}{\partial v_l} > 0,\ l \in \mathcal{L}$, and $\frac{\partial g_k(\rho^*, \pi_k^*)}{\partial \pi_k} < 0,\ k \in \mathcal{C}$, holds. Then, the values of the link flow and demand perturbation v' and d' are unique; therefore, the value v' (respectively, d') is the directional derivative of the equilibrium link flow (respectively, demand), at ρ^*, in the direction of ρ'.
- The link flow solution v^* is differentiable at ρ^* if and only if the following statement holds for every $r \in \mathcal{R}$: if $h_r^* = 0$ on the set of equilibria then $h'_r = 0$ on the set of sensitivity solutions, for every choice of $\rho' \in \Re^d$.

Notice that if d is fixed, then the uniqueness of t', c', π', and v' follow without making reference to any of the properties of g above. Notice further that the differentiability characterization is strictly milder than the strict complementarity of h^*. The results for more general cases are slightly weaker, due to the slightly stronger requirements for the TEP solution entities to be unique.

The case of the logit-based TEP is interesting. Adding the term $\frac{1}{\theta}\sum_{r\in\mathcal{R}} h_r \log h_r$ to the TEP, we notice that $h^* > 0^{|\mathcal{R}|}$ will always hold. Therefore, the logit SUE solution (h^*, v^*) is always differentiable. Further, if we let $\theta \to \infty$ then the limit of the sequence of SUE solution gradients of v^* is the gradient of v^* in the UE solution if and only if the limit exists!

5 Limitations of a popular sensitivity analysis tool

The analysis in [ToF88] is performed on a a fixed demand problem. The first condition ensures local uniqueness:

(Condition 1—strong monotonicity) $t(\rho, \cdot)$ is strongly monotone in a neighbourhood of ρ^*.

This condition is stronger than necessary, as we have seen.

The analysis is based on first selecting a particular equilibrium route flow solution. Among the conditions stated, the route flow is supposed to be strictly complementary. The common definition of *strict complementarity* of a route flow solution h^* is that it is complementary (that is, that $0 \leq h_r^* \perp [c_r(\rho^*, h^*) - \pi_{pq}(\rho^*)] \geq 0$ holds for all $r \in \mathcal{R}_{pq}$, $(p, q) \in \mathcal{C}$), and

$$h_r^* + [c_r(\rho^*, h^*) - \pi_{pq}(\rho^*)] > 0, \qquad r \in \mathcal{R}_{pq}, \quad (p, q) \in \mathcal{C}.$$

(An arbitrary route $r \in \mathcal{R}_{pq}$ is either used or it is more expensive than the routes used in the OD pair.) Tobin and Friesz state a definition of traffic equilibrium in terms of *total* link flows v only. The problem with this definition is that they work with *aggregated* potential differences $\lambda_j^* - \lambda_i^*$, which may not exist:

(Condition 2—strict complementarity) For each link $l = (i, j) \in \mathcal{L}$, $v_l^* = 0 \implies t_l(\rho^*, v^*) > \lambda_j^* - \lambda_i^*$ holds.

Whenever the total link flow vector v^* is positive, this condition is satisfied. Clearly, it is not compatible with the strict complementarity condition (5).

Next, we are asked to restrict the network $\mathcal{G}$ to $\mathcal{G}_+ = (\mathcal{N}, \mathcal{L}_+)$, where $l \in \mathcal{L}_+$ if and only if $v_l^* > 0$, that is, to the network corresponding to the links having a positive flow given ρ^*. Consequently, there are possibly some routes that will be removed as well. The + notation to follow reflects this restriction. Under the assumptions stated sofar, the set $H_+^*(\rho^*)$ of equilibrium route flows is a bounded polyhedron. The next condition states that an equilibrium route flow vector h_+^* is selected such that it is a "non-degenerate extreme point" of $H_+^*(\rho^*)$:

(Condition 3—linear independence) An equilibrium route flow h_+^* is chosen such that it is an extreme point of $H_+^*(\rho^*)$ which has exactly as many routes with a positive flow as the rank of the matrix $[\Lambda_+^{\mathrm{T}} \mid \Gamma_+]$.

The rank of this matrix is not higher than the number of links with a positive flow at v^* plus $|\mathcal{C}|$. An LP is stated for generating such a point, but the sensitivity values do not depend on this choice as long as it is an extreme point of $H_+^*(\rho^*)$.

A final restriction is then made, such that we remove all the indices in the vector h_+^* for which the flow is zero. (We do not change the notation to reflect this restriction.) The sensitivity problem is then finally set up as follows:

$$\begin{pmatrix} \nabla_\rho h_+ \\ \nabla_\rho \pi \end{pmatrix} = \begin{pmatrix} \nabla_h c_+(\rho^*, h_+^*) & -\Lambda_+^{\mathrm{T}} \\ \Lambda_+ & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\nabla_\rho c_+(\rho^*, h_+^*) \\ \nabla_\rho g(\rho^*) \end{pmatrix}. \tag{2}$$

A case of differentiability where the formula (2) fails Consider the network shown in Figure 1.

There is a single OD pair, $(1, 3)$, with a fixed demand of 2 units of flow. The link cost functions are given by

$$t_1(v_1, \rho) := v_1 + \rho; \quad t_2(v_2) := v_2; \quad t_3(v_3) := v_3; \quad t_4(v_4) := v_4.$$

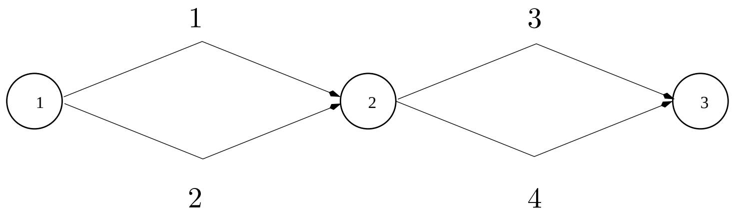

Figure 1: Network for the first counter-example.

We have four routes: $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, and $\{2,4\}$.

With $\rho^* = 0$, the unperturbed traffic equilibrium solution is $v^* = (1,1,1,1)^{\mathrm{T}}$. The solution is differentiable, since it is even strictly complementary. The derivative with respect to ρ at ρ^* is $(-\frac{1}{2}, 0, \frac{1}{2}, 0)^{\mathrm{T}}$, which is quite intuitive.

For the formula (2), we fulfill Conditions 1 and 2, because $v^* > 0^{|\mathcal{L}|}$ and $\mathcal{G}_+ = \mathcal{G}$. We last try to comply with the linear independence Condition 3, by choosing the right equilibrium route flow solution. Note then that

$$[\Lambda^{\mathrm{T}} \mid \Gamma] = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

which has rank 3. So, we should find a route flow solution, h^*, in which exactly 3 routes have a positive flow. This is however impossible; the only alternatives are 2 or 4. To see why, let's suppose that the flow on the first route, $\{1,3\}$, is $\alpha \in [0,1]$. Then, the flows on the routes $\{1,4\}$ and $\{2,3\}$ must both be $1-\alpha$, in order to comply with the total flow on the links. This implies that the flow on route $\{2,4\}$ is α. This shows that for any value of $\alpha \in [0,1]$, the number of routes having a non-zero flow is either 2 or 4. We can therefore not comply with Condition 3, and the formula fails, even though the gradient exists. We remark that there are differentiable points that are even non-strictly complementary.

References

[CSF00] H.-J. Cho, T. E. Smith, and T. L. Friesz, "A reduction method for local sensitivity analyses of network equilibrium arc flows," *Transportation Research*, 34B (2000), pp. 31–51.

[DoR01] A.L. Dontchev and R.T. Rockafellar, "Ample parameterization of variational inclusions," SIAM Journal on Optimization, 12 (2001), pp. 170–187.

[JoP03] M. Josefsson and M. Patriksson, "Sensitivity analysis of separable traffic equilibria, with application to bilevel optimization in network design," submitted to *Transportation Research, B.*

[Kyp90] J. Kyparisis, "Solution differentiability for variational inequalities," *Mathematical Programming*, 48 (1990), pp. 285–301.

[Pat94] M. Patriksson, *The Traffic Assignment Problem—Models and Methods*, Topics in Transportation, VSP BV, Utrecht, The Netherlands, 1994.

[Pat04] M. Patriksson, "Sensitivity analysis of traffic equilibria," *Transportation Science*, 38 (2004), to appear.

[Pat05] M. Patriksson, *Traffic Equilibrium Models: Analysis, Applications, and Optimization Algorithms*, Springer-Verlag, Berlin, under construction.

[PaR03] M. Patriksson and R. T. Rockafellar, "Sensitivity analysis of variational inequalities over aggregated polyhedra, with application to traffic equilibria," *Transportation Science*, 37 (2003), pp. 56–68.

[ToF88] R. L. Tobin and T. L. Friesz, "Sensitivity analysis for equilibrium network flow," *Transportation Science*, 22 (1988), pp. 242–250.