Modeling the Dynamic Formation of Distribution Networks and Supply Chains Using Self-Regulating Variational Inequalities

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1 Introduction

A number of models of dynamic oligopolistic network competition have been reported in the regional science and spatial economics literature. In this paper we show how such models of interest may be extended to create descriptive dynamic models of freight networks and supply chains using new results concerning differential variational inequalities.

Here we take the point of view that input factors for supply chains and freight network services for output distribution selected by firms in their business relationships may only be accurately modelled by considering the combined supply-production-distribution policies of those firms, as they compete with one another via a generalized multi-layer transport network. We assume that the output market for the firms is organized as a non-cooperative spatial oligopoly. Production schedules and physical distribution plans for finished goods are determined by dynamic oligopolistic competition among their producers. To model supply chains, we assume that input factor prices are set by contracts extending over the planning horizon and that just-in-time use of factor inputs is pervasive. The exercise of these contracts is also governed by dynamic oligopolistic competition among the producers of finished goods. The models presented are strategic in nature, meaning that they determine the broad outline of optimal policies for the efficient formation and use of distribution networks and supply chains when these are disaggregated by input factor type and characterized as time-varying flows. The foundation model from which the models presented in this paper are derived is a recent formulation of dynamic oligopolistic network competition reported by Friesz *et al* [1]. We extend this foundation model to include an explicit network for input factor (supply chain) flows as well as a network for distribution (freight) flows. We also introduce factor sequence restrictions, which are constraints that enforce a desired arrival order for the factor inputs. The resulting model is a special type of generalized differential game that can be expressed as an infinite dimensional variational inequality with state-dependent time lags.

2 Infinite Dimensional Variational Inequalities

The models of this paper involve mappings between Hilbert spaces[1]. The specific Hilbert spaces we employ in our exposition are those that allow optimal control problems to be analyzed as infinite dimensional mathematical programs, and are chosen here since we are extending the notion of an optimal control problem to the more general setting of an infinite dimensional variational inequality.

We begin by letting

$$\begin{aligned} u &\in \left(L_+^2\left[t_0, t_F\right]\right)^m \\ x\left(u,t\right) &= \arg\left\{\frac{dy}{dt} = f\left(y,u,t\right),\ y\left(t_0\right) = y^0,\ \Gamma\left[y\left(t_F\right), t_F\right] = 0\right\} \in \left(\mathcal{H}^1\left[t_0, t_F\right]\right)^n \end{aligned} \tag{1}$$

The entity $x\left(u,t\right)$ is to be interpreted as an operator that tells us the state variable x for each vector u and each time $t \in [t_0, t_F] \subseteq \Re_+^1$; constraints on u are enforced separately [2]. The variational inequality of interest to us in this paper takes the form:

$$\begin{gathered} \text{find } u^* \in U \text{ such that} \\ \left\langle G\left(x\left(u^*,t\right), u^*, t\right), u - u^*\right\rangle \geq 0 \text{ for all } u \in U \end{gathered} \tag{2}$$

where

$$U \subseteq \left(L_+^2\left[t_0, t_F\right]\right)^m \tag{3}$$

$$x^0 \in \Re^n \tag{4}$$

$$G : \left(\mathcal{H}^1\left[t_0, t_F\right]\right)^n \times \left(L_+^2\left[t_0, t_F\right]\right)^m \times \Re_+^1 \longrightarrow \left(L_+^2\left[t_0, t_F\right]\right)^m \tag{5}$$

$$f : \left(\mathcal{H}^1\left[t_0, t_F\right]\right)^n \times \left(L_+^2\left[t_0, t_F\right]\right)^m \times \Re_+^1 \longrightarrow \left(L_+^2\left[t_0, t_F\right]\right)^n \tag{6}$$

$$\Gamma : \left(\mathcal{H}^1\left[t_0, t_F\right]\right)^n \times \Re_+^1 \longrightarrow \left(\mathcal{H}^1\left[t_0, t_F\right]\right)^r \tag{7}$$

Note that $\left(L_+^2\left[t_0, t_F\right]\right)^m$ is the m-fold product of the space of non-negative square-integrable functions $L_+^2\left[t_0, t_F\right]$ defined on the segment of the real line $[t_0, t_F] \in \Re_+^1$ and the inner product

[1] A Hilbert space is Banach space with a well-defined inner product that induces a norm; a Banach space is a complete vector space with a well defined norm.

[2] This definition of $x\left(u,t\right)$ follows that given ofy Minoux (Chapter 10) [2] for analyzing optimal control problems from the point of view of infinite dimensional mathematical programming. Moreover, unless other conditions are satisfied $x\left(u,t\right)$ is not a solution of the variational inequality considered in (2); rather it should be thought of as a parametric representation of the state vector in terms of the controls. Note also that we do not actually have to explicitly solve for $x(u,t)$, as is made clear in our subsequent analysis.

in (2) is defined by

$$\langle G\left(x\left(u^{*},t\right),u^{*},t\right),u-u^{*}\rangle \equiv \int_{t_0}^{t_F}\left[G\left(x\left(u^{*},t\right),u^{*},t\right)\right]^{T}\left(u-u^{*}\right)\geq 0$$

while $\left(\mathcal{H}^1\left[t_0,t_F\right]\right)^n$is the n-fold product of the Sobolev space $\mathcal{H}^1\left[t_0,t_F\right]$ defined on the segment of the real line $[t_0,t_F]\in\Re^1_+$. We refer to (2) as $SRVI(F,f,U)$. It has been shown formally in Friesz *et al* [1] that, under mild regularity conditions, any problem of the above type may be recast as a nonlinear complementarity problem (NCP) and solved using time discretization.

3 Constrained Dynamics of Supply Chain Evolution

In what follows we describe how supply networks may be modeled as self-regulating variational inequalities and solved to describe the evolution over time of supply chains. We employ a supply network which is connected to the production process of the firm of interest at supply-intake nodes. For the supply subnetwork origin nodes are the sources of factor supplies while destination nodes represent the location in time and space at which factors enter the production process. The production process will typically involve several stages, and as such is described by paths through a production network whose nodes are the various stages of production. Although this perspective is reminiscent of the well known critical path method, it is substantially more general. We assume that the supply-production network just described has associated demands for finished goods that compel production activity that in turn compels the formation supply chains within the supply subnetwork. Of course the supply and production processes we consider are dynamic. Kachani *et al* [3] described a fluid model of the dynamic pricing and inventory management for make-to-stock manufacturing systems considering the fact that a unit of a product incurs a delay before being sold and that delay is similar to the travel times incurred in a transportation network. This same perspective is employed in our model of supply chain evolution presented in this paper.

3.1 Notation for the Supply Network

We will be treating the time varying flows of production factors over a network associated with the graph $G_s\left(\mathcal{N},\mathcal{A}\right)$ where $\mathcal{N}$ is the relevant set of nodes and $\mathcal{A}$ is the relevant set of arcs. To model such flows, we take a path $p\in\mathcal{P}^k$ of the supply network to be a sequence of arcs labeled as follows:

$$p\doteq\left\{a_1,a_2,...,a_{i-1},a_i,a_{i+1},...,a_{m(p)}\right\}$$

where $\mathcal{P}^k$ is the set of all paths associated with the supply of production factor $k\in\mathcal{K}$ and $\mathcal{K}$ is the set of all production factors.

It will also be expedient to let the set of all supply-intake pairs pertinent to factor $k\in\mathcal{K}$ and to firm $f\in\mathcal{F}$ be $\mathcal{W}_s^{kf}$. Moreover, $\mathcal{P}_{ij}^{kf}$ will denote the set of paths for movement of factor $k\in\mathcal{K}$ between supply-intake pair $(i,j)\in\mathcal{W}_s^{kf}$ associated with firm $f\in\mathcal{F}$. For a given path, $p\in\mathcal{P}_{ij}^{kf}$, the tail node of arc a_1 is the source of factor $k\in\mathcal{K}$ intended for use by firm $f\in\mathcal{F}$, while the head node of arc $a_{m(p)}$ is the location to which that same factor is delivered to the that firm. Where necessary we will denote the set of all paths pertinent to factor $k \in \mathcal{K}$ uses by firms $f \in \mathcal{F}$ by $\mathcal{P}^{kf}$.

We denote the set of supply origin nodes for firm $f \in \mathcal{F}$ by $\mathcal{N}_O^{kf}$ and the set of destination (intake) nodes for firm $f \in \mathcal{F}$ by $\mathcal{N}_D^f$, assuming that an intake node may accommodate all factors of production. Each arc traversed on the way between $(i,j) \in \mathcal{W}_s^{kf}$ represents either physical transportation or required pre-processing of factor $k \in \mathcal{K}$ originating at node $i \in \mathcal{N}_O^{kf}$ prior to delivery at intake node $j \in \mathcal{N}_D^f$.

We also let $\tau_{a_i}^{pk}$ be the time of exit of flow from arc $a_i \in p$ given shipment of factor $k \in \mathcal{K}$ at time t via path $p \in \mathcal{P}^{kf}$. Furthermore we take

$$\delta_{a_i p} = \begin{cases} 1 & \text{if} \quad a_i \in p \\ 0 & \text{if} \quad a_i \notin p \end{cases}$$

to be an element of the arc-path incidence matrix.

3.2 The Supply Network Dynamics and Constraints

The relevant arc dynamics for factors supplied to the producing firm $f \in \mathcal{F}$ are

$$\begin{aligned} \frac{dx_{a_i}^{pk}(t)}{dt} &= g_{a_{i-1}}^{pk}(t) - g_{a_i}^{pk}(t) \qquad \forall k \in \mathcal{K}, f \in \mathcal{F}, p \in \mathcal{P}^{kf}, i \in [1, m(p)] && (8) \\ x_{a_i}^{pk}(0) &= x_{a_i,0}^{pk} \qquad \forall k \in \mathcal{K}, f \in \mathcal{F}, p \in \mathcal{P}^{kf}, i \in [1, m(p)] && (9) \end{aligned}$$

where $x_{a_i}^{pk}$ is the volume of factor k on arc a_i, $g_{a_i}^{pk}$ is flow of that factor exiting arc a_i and $g_{a_{i-1}}^{pk}$ is flow the same factor entering arc a_i of path $p \in \mathcal{P}^k$. Also, $g_{a_0}^{pk}$ is the flow exiting the origin of path p. These notational conventions mean that each path carries a firm-specific identity; hence in specifying a path one is also specifying a firm. Furthermore

$$x_a(t) = \sum_{f \in \mathcal{F}} \sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}^{kf}} \delta_{ap} x_a^{pk}(t) \quad \forall a \in \mathcal{A} \tag{10}$$

is the total arc volume[3].

Of course total path traversal time is

$$D_p^k(t) = \sum_{i=1}^{m(p)} \left[\tau_{a_i}^{pk}(t) - \tau_{a_i-1}^{pk}(t) \right] = \tau_{a_{m(p)}}^{pk}(t) - t \quad \forall f \in \mathcal{F}, k \in \mathcal{K}, p \in \mathcal{P}^{kf}$$

since we use the convention that

$$\tau_0^{pk}(t) = t$$

It will prove expedient to introduce the following recursive relationships that must hold in light of the above development:

$$\tau_{a_1}^{pk} = t + D_{a_1}^k(t) \quad \forall f \in \mathcal{F}, k \in \mathcal{K}, p \in \mathcal{P}^{kf}$$

[3]In expression (10) we assume that all factor flows are in comensurable units, such as pounds per second. However, it is not difficult to relax this assumption and use explicit factor weights to convert factor flows to common units of measurement.

$$\tau_{a_i}^{pk}(t) = \tau_{a_{i-1}}^{pk}(t) + D_{a_i}^{k}(\tau_{a_{i-1}}^{p}) \quad \forall f \in \mathcal{F}, k \in \mathcal{K}, p \in \mathcal{P}^{kf}, i \in [2, m(p)]$$

where $D_{a_i}^{pk}(t)$ for $i \in [1, m(p)]$ is the traversal time for an arbitrary supply network arc $a_i \in p \in \mathcal{P}^{kf}$. The dependence of arc traversal time on clock time t reflects the fluctuations of resources needed for transportation or preprocessing factors as they migrate through the supply network. We may easily ensure that a first-in-first out queue discipline is enforced by imposing the constraint

$$\frac{d}{dt}\left[t + D_{a_i}(t)\right] = 1 + \frac{d}{dt} D_{a_i}(t) > 0$$

for an arbitrary supply network arc $a_i \in p \in \mathcal{P}^{k}$. This constraint assures that each exit time function is strictly monotonically increasing during periods when flow is strictly positive. Furthermore, physical reality and elementary manipulations based on the chain rule lead to the following flow propagation constraints:

$$g_{a_i}^{pk}\left(t + D_{a_i}^{k}(t)\right) \cdot \left(1 + \frac{d}{dt} D_{a_i}^{k}(t)\right) = g_{a_{i-1}}^{pk}(t) \quad \forall f \in \mathcal{F}, k \in \mathcal{K}, p \in \mathcal{P}^{kf}, i \in [0, m(p)] \tag{11}$$

These are *proper flow progression constraints* derived in a fashion that makes them completely *consistent with the chosen arc dynamics and model of arc delay.* These constraints clearly involve time-dependent time shifts $D_{a_i}^{k}(t)$ for each supply network arc $a_i \in p \in \mathcal{P}^{kf}$.

To discourage the early/late arrival of input factors of production and thereby ensure realistic behavior, we employ asymmetric early/late arrival penalties[4]

$$\Phi_p^k\left[t + D_p^k(x,t) - \chi_{kf}\right] \tag{12}$$

where χ_{kf} is the desired arrival time for shipments of factor $k \in \mathcal{K}$ to firm $f \in \mathcal{F}$ carried via path $p \in \mathcal{P}^{kf}$. We also define a unit delivery fee $r_{ij}^{k}(t)$ for each $f \in \mathcal{F}, k \in \mathcal{K}, i \in \mathcal{N}_O^{kf}, j \in \mathcal{N}_D^{f}$. We combine the actual path delays and arrival penalties to obtain

$$\Psi_p^k(t) = r_{ij}(t) + D_p^k(t) + \Phi_p^k\left[t + C_p^k(t) - \chi_{kf}\right]$$

$$\forall f \in \mathcal{F}, k \in \mathcal{K}, i \in \mathcal{N}_O^{kf}, j \in \mathcal{N}_D^{f}, p \in \mathcal{P}_{ij}^{kf}$$

which we call the *effective delay operators.*

We assume there is an exogenous instantaneous flow demand so that flow conservation becomes

$$z_{ij}^{kf}(t) = \sum_{p \in \mathcal{P}_{ij}^{kf}} g_{a_{m(p)}}^{pk}(t) \ \forall k \in \mathcal{K}, f \in \mathcal{F}, i \in \mathcal{N}_D^{f}, j \in \mathcal{N}_O^{kf} \tag{13}$$

where z_{ij}^{kf} is the demand of producing firm $f \in \mathcal{F}$ to have factor $k \in \mathcal{K}$ available at intake node $i \in \mathcal{N}_D^{f}$ from the source node $j \in \mathcal{N}_O^{kf}$ with the ideal arrival time being χ_{kf}. Arrival before or after χ_{kf} will generally occur only if feasibility or cost considerations require so, as there is a strictly positive penalty (12) for missing the targeted delivery time. Naturally we impose the nonnegativity restrictions

$$x \geq 0 \qquad g \geq 0 \qquad h \geq 0 \tag{14}$$

[4]See Friesz *et al* [4] for more detail about such penalties.

where

$$x \equiv \left(x_{a_i}^{pk} : k \in \mathcal{K}, f \in \mathcal{F}, p \in \mathcal{P}^{kf}, i \in [1, m(p)]\right) \tag{15}$$

$$g \equiv \left(g_{a_i}^{pk} : k \in \mathcal{K}, f \in \mathcal{F}, p \in \mathcal{P}^{kf}, i \in [0, m(p)]\right) \tag{16}$$

are the relevant vectors of state variables and control variables. We will also use the notation

$$x^0 = \left(x_{a_i,0}^{pk}(0) : k \in \mathcal{K}, f \in \mathcal{F}, p \in \mathcal{P}^{kf}, i \in [1, m(p)]\right)$$

to refer to the vector of initial state values. We may now define

$$\Omega = \{g \geq 0 : (11), (13) \text{ and } (14) \text{ hold}\} \tag{17}$$

which is the set of control variables that represent physically meaningful factor supply flows.

As a consequence of the preceding notation and development, we may state the constrained dynamics for our supply network model as:

$$\frac{dx_{a_i}^{pk}(t)}{dt} = g_{a_{i-1}}^{pk}(t) - g_{a_i}^{pk}(t) \qquad \forall\, k \in \mathcal{K}, f \in \mathcal{F}, p \in \mathcal{P}^{kf}, i \in [1, m(p)] \tag{18}$$

$$g \in \Omega \tag{19}$$

$$x(0) = x^0 \tag{20}$$

which makes clear that the link volumes $x_{a_i}^{pk}$ are natural state variables while the link entrance (exit) flows $g_{a_i}^{pk}$ are natural control variables in our formulation.

4 A Combined Model of Supply, Production and Distribution

In this section we model oligopolistic competition among spatially separated producers of a single good sharing the same distribution infrastructure when those producers are informed by the just-in-time system for supply of input factors described in Section 3.

4.1 Distribution Network Notation and Assumptions

We will need the following additional notation/assumptions to model the oligopolistic firms sharing factor inventory suppliers and distribution network infrastructure:

1. *Distribution network.* The distribution network is separate from the factor supply network and flows on one do not induce congestion on the other. The graph underlying the distribution network is denoted by $G_d(\mathcal{M}, \mathcal{B})$, where $\mathcal{M}$ is the set of markets at which production, consumption and or transshipment occurs and $\mathcal{B}$ is the set of arcs that connect those markets and create a network economy. We work with a fairly general setting here with the following features :

- Each firm competes in all markets, $i \in \mathcal{M}$;
- Firm $f \in \mathcal{F}$ is located at nodes $j \in \mathcal{N}_D^f$ with generally, $\mathcal{N}_D^f \cap \mathcal{M} \neq \mathcal{M}$
- Firm $f \in \mathcal{F}$ is capable of shipping finished goods from node $i \in \mathcal{N}_D^f$ to all the nodes $j \in \left(\mathcal{N}_D^f \cup \mathcal{M}\right) \backslash i$.

2. *Value of time.* We let v_{kf} denote the constant value of time for firm $f \in \mathcal{F}$ and input factor $k \in \mathcal{K}$.

3. *Oligopolistic competition in the output market.* The producing firm is one of several that are spatially separated and located at nodes of the underlying graph. For each $i \in \mathcal{M}$, the inverse demand functions $\pi_i\left(d_i, t\right)$ are known, where $\mathcal{M}$ is the set of all markets at which the firm is produced and/or consumed, and aggregate demand d_i for the finished good at $i \in \mathcal{M}$. That aggregate demand obeys for all $i \in \mathcal{M}$ the relationship

$$d_i = \sum_{f \in \mathcal{F}} d_i^f$$

where d_i^f is the allocation of to output of firm $f \in \mathcal{F}$ to demand at market i.

4. *Variable production cost.* Production cost of the firm consists of two components : a fixed cost and a variable cost, depending on the flow rate of each of the input factors, $k \in \mathcal{K}$. We define

$$\begin{aligned} z_{ij}^{kf}(t) &= \sum_{p \in \mathcal{P}_{ij}^{kf}} g_{a_{m(p)}}^{pk}(t) \qquad \forall k \in K, f \in F, i \in N_D^f, j \in N_O^{kf} \\ z_i^f &= \left(z_{ij}^{kf} : k \in \mathcal{K}; j \in \mathcal{N}_O^{kf}\right) \quad \forall f \in \mathcal{F}, i \in \mathcal{N}_D^f \end{aligned}$$

where $z_{ij}^{kf}(t)$ is the flow rate of input factor $k \in \mathcal{K}$ from source node $j \in \mathcal{N}_O^{kf}$ to the factor intake node $i \in \mathcal{N}_D^f$ for firm $f \in \mathcal{F}$. Furthermore, ξ_j^{kf} is the price of the input factor $k \in \mathcal{K}$ when purchased by firm $f \in \mathcal{F}$ from the supplier at node $j \in \mathcal{N}_O^{kf}$. Letting

$$\xi^f = \left(\xi_j^{kf} : k \in \mathcal{K}, j \in \mathcal{N}_O^{kf}\right)$$

Therefore the total cost of input factors for firm $f \in \mathcal{F}$ is

$$b^f + \sum_{i \in \mathcal{N}_D^f} \left(\xi^f\right)^T z_i^f$$

where b_j^{kf} is the fixed price of the input factor $k \in \mathcal{K}$ when purchased by firm $f \in \mathcal{F}$ from the supplier at node $j \in \mathcal{N}_O^{kf}$ and

$$b^f = \left(b_j^{kf} : k \in \mathcal{K}, j \in \mathcal{N}_O^{kf}\right)$$

5. *Production function and revenue.* Each firm $f \in \mathcal{F}$ at each market $i \in \mathcal{M}$ has a production function $F_i^f\left(z^f\right)$, which when multiplied by inverse demand yields total revenue.

6. *Inventory holding cost.* Each firm $f \in \mathcal{F}$ holds inventories of finished goods at its facility-nodes as well as at each markets (even though the firm doesn't have production facilities there), $i \in \left(\mathcal{N}_D^f \cup \mathcal{M}\right)$ and is accountable for the inventory holding cost at their location which is computed from the function
$$\phi_i^f\left(I_i^f, t\right) \quad \forall f \in \mathcal{F}, i \in \left(\mathcal{N}_D^f \cup \mathcal{M}\right)$$
where I_i^f is total inventory for the market and firm of interest.

7. *Shipment cost.* We let $r_p^f(t) \in \Re_{++}^1$ be the freight rate (tariff) charged to firm $f \in \mathcal{F}$ per unit of flow h_p^f for path $p \in \mathcal{S}^f$ of the distribution network, where $\mathcal{S}^f$ denotes the set of all such paths :
$$\mathcal{S}^f = \left\{(i,j) : i \in \mathcal{N}_D^f, j \in \left(\mathcal{N}_D^f \cup \mathcal{M}\right) \backslash i\right\}$$

8. *Inventory dynamics.* Each producing firm $f \in \mathcal{F}$ at each node $i \in \left(\mathcal{N}_D^f \cup \mathcal{M}\right)$ obeys the flow conservation constraint which forms the state dynamics. In the case when a given market is also a production site for firm $f \in \mathcal{F}$ at some node $i \in \left(\mathcal{N}_D^f \cap \mathcal{M}\right)$
$$\frac{dI_i^f(t)}{dt} = F_i^f\left(z^f\right) + \sum_{j \in \left\{\mathcal{N}_D^f \cup \mathcal{M} \backslash i\right\}} \sum_{p \in \mathcal{S}_{ji}^f} h_p^f - \sum_{j \in \left\{\mathcal{N}_D^f \cup \mathcal{M} \backslash i\right\}} \sum_{p \in \mathcal{S}_{ij}^f} h_p^f - d_i^f \;\; \forall i \in \left(\mathcal{N}_D^f \cap \mathcal{M}\right) \tag{21}$$
For some other nodes $i \in \left(\mathcal{N}_D^f \backslash \mathcal{M}\right)$, where a given production site for firm $f \in \mathcal{F}$ does not have a market, the inventory dynamics follow
$$\frac{dI_i^f(t)}{dt} = F_i^f\left(z^f\right) + \sum_{j \in \left\{\mathcal{N}_D^f \cup \mathcal{M} \backslash i\right\}} \sum_{p \in \mathcal{S}_{ji}^f} h_p^f - \sum_{j \in \left\{\mathcal{N}_D^f \cup \mathcal{M} \backslash i\right\}} \sum_{p \in \mathcal{S}_{ij}^f} h_p^f \;\; \forall i \in \left(\mathcal{N}_D^f \backslash \mathcal{M}\right) \tag{22}$$
Similarly, when a given market is not a production site for firm $f \in \mathcal{F}$ at some node $i \in \left(\mathcal{M} \backslash \mathcal{N}_D^f\right)$
$$\frac{dI_i^f(t)}{dt} = \sum_{j \in \left\{\mathcal{N}_D^f \cup \mathcal{M} \backslash i\right\}} \sum_{p \in \mathcal{S}_{ji}^f} h_p^f - \sum_{j \in \left\{\mathcal{N}_D^f \cup \mathcal{M} \backslash i\right\}} \sum_{p \in \mathcal{S}_{ij}^f} h_p^f - d_i^f \;\; \forall i \in \left(\mathcal{M} \backslash \mathcal{N}_D^f\right) \tag{23}$$
Appropriate initial conditions are
$$I_i^f(0) = K_i^f \;\; \forall f \in \mathcal{F}, i \in \left(\mathcal{N}_D^f \cup \mathcal{M}\right) \tag{24}$$

9. *Factor sequencing constraints.* To understand these constraints, consider the case when, at intake node $i \in \mathcal{N}_D^{kf}$, factor $k \in \mathcal{K}$ must arrive a constant $\Delta \in \Re_{++}^1$ units of time in advance of another factor, say $\ell \in \mathcal{K}$, and its flow must be at least a constant $M \in \Re_{++}^1$ times greater. We express this constraint as
$$z_i^{kf}(t) \geq M \cdot z_i^{\ell f}(t + \Delta) \quad \forall (i, k, \ell, f) \in \Lambda \tag{25}$$
where Λ is the set of all node-factor-factor-firm 4-tuples for which such restrictions apply. Clearly, for complete generality the time advance Δ and constant M would differ from firm-to-firm as well as from node-to-node and factor–pair-to-factor-pair. However, we suppress that generality as it involves only more complex notation without providing additional insight.

10. *Bounds on controls.* All demand and shipping variables for all firms $f \in \mathcal{F}$ are non-negative and bounded from above; that is

$$
\begin{aligned}
0 &\leq d_i^f \leq D_i^f \quad \forall i \in \mathcal{M} && (26)\\
0 &\leq h_p^f \leq H_p^f \quad \forall p \in \mathcal{S}^f && (27)
\end{aligned}
$$

4.2 The Extremal Problem for an Individual Firm

We are now in a position to completely describe the optimal control problem for a single producing firm $f \in \mathcal{F}$:

$$
\begin{aligned}
J_f &= \int_0^T \exp(-\rho t) \left\{ \sum_{i\in\mathcal{M}} \pi_i \left(\sum_{m\in\mathcal{F}} d_i^m, t \right) F_i^f\left(z^f\right) - \sum_{i\in\mathcal{N}_D^f} b^f + \left(\xi^f\right)^T \cdot z_i^f - \sum_{i\in\mathcal{N}_D^f\cup\mathcal{M}} \phi_i^f\left(I_i^f, t\right) \right\} dt \\
&\quad - \int_0^T \exp(-\rho t) \left\{ \sum_{j\in\left\{\mathcal{N}_D^f\cup\mathcal{M}\setminus i\right\}} \sum_{p\in S_{ij}^f} r_p^f(t) h_p^f(t) + \sum_{k\in\mathcal{K}} \sum_{p\in\mathcal{P}^{kf}} v_k \Psi_p^k(t)\, g_{a_{m(p)}}^p(t) \right\} dt \qquad (28)
\end{aligned}
$$

subject to (11), (13), (14), (21), (22), (23), (24), (25), (26) and (27).

4.3 The SRVI Formulation

Before we proceed further, we need to define the following vector notations for the controls and states applied to the optimal control problem (28). Let us define

$$
\begin{aligned}
d^f &= \left(d_i^f : i \in \mathcal{M}\right) && (29)\\
h^f &= \left(h_p^f : p \in \mathcal{S}^f\right) && (30)\\
g^f &= \left(g_{a_i}^{kf} : k \in \mathcal{K}, p \in \mathcal{S}^f, i \in [0, m(p)]\right) && (31)
\end{aligned}
$$

Let Π_f denote the feasible set of the controls for firm by $f \in \mathcal{F}$. Then each firm faces the problem

$$
\left.
\begin{array}{l}
\max \ J_f \\
\text{subject to } \left(d^f, h^f, g^f\right) \in \Pi_f
\end{array}
\right\} \forall f \in \mathcal{F} \qquad (32)
$$

Note that (32) defines a Cournot-Nash game expressed as a set of coupled optimal control problems, one for each firm $f \in \mathcal{F}$. It can be demonstrated formally that solutions of the following SRVI, when they exist, are Cournot-Nash (CN) equilibria for the above game :

$$
\text{find } \left(d^{*f}, h^{*f}, g^{*f}\right) \in \Pi \text{ such that}
$$

$$
0 \geq \sum_{f\in\mathcal{F}} \int_0^T \left[\begin{array}{c} \sum_{i\in\mathcal{M}} \frac{\partial \tilde{H}_f^*}{\partial d_i^f} \left(d_i^f - d_i^{*f}\right) + \sum_{p\in\mathcal{S}^f} \frac{\partial \tilde{H}_f^*}{\partial h_p^f} \left(h_p^f - h_p^{*f}\right) + \\ \sum_{k\in\mathcal{K}} \sum_{p\in\mathcal{P}^{kf}} \sum_{i=1}^{m(p)} \frac{\partial \tilde{H}_f^*}{\partial g_{a_i,p}^{kf}} \left(g_{a_i,p}^{kf} - g_{a_i,p}^{*kf}\right) \end{array} \right] dt \qquad (33)
$$

$$\text{for all } \left(d^f, h^f, g^f\right) \in \Pi$$

where $\tilde{H}_f$ is the Hamiltonian formed from the optimal control problem (32) for all $f \in \mathcal{F}$. This SRVI can be recast as functional math program (for more details please refer [1]) and solved using commercial nonlinear solver.

5 Agent-based Simulation Framework

We do not yet know whether the NCP/finite element approach will prove practical for medium and large size problems. Accordingly, we are also exploring an agent based simulation approach that captures the decision processes of firms as we have modeled them in section 4. It is our present intention to include results and comparisons of the NCP/finite element and ABS approaches in the final paper to be presented at TRISTAN V.

6 Conclusions and Future Work

Numerical examples of the combined supply-production-distribution model are presently being prepared and it is our aspiration to present those results at TRISTAN V.

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