A New Branch-and-Cut Algorithm for the Undirected m-Peripatetic Salesman Problem

Eric Duchenne* Gilbert Laporte† Frédéric Semet*

*LAMIH-ROI, Valenciennes University
Le Mont-Houy
59313 Valenciennes, France
{eric.duchenne,frederic.semet}@univ-valenciennes.fr

†GERAD, Canada Research Chair in Distribution Management and HEC Montréal
3000 chemin de la Côte-Sainte-Catherine
Montreal, Canada H3T 2A7
gilbert@crt.umontreal.ca

1 Introduction

The m-Peripatetic Salesman Problem (m-PSP) is defined on a complete graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ is a vertex set and $E = \{(i, j)\colon\ i, j \in V,\ i < j\}$ is an edge set. A cost matrix $C = (c_{ij})$ is defined on E. The problem consists of determining m edge disjoint Hamiltonian cycles of minimum total cost on G. When $m = 1$ the m-PSP reduces to the *Traveling Salesman Problem* (TSP). In the sequel we assume that $m < (n-1)/2$ to avoid trivial or infeasible cases.

The m-PSP was introduced by Krarup (1975). Applications include the design of watchman tours (Wolfter Calvo and Cordone, 2003) where it is often important to assign a set of edge-disjoint rounds to the watchman in order to avoid always repeating the same tour and thus enhance security. In the same spirit, De Kort (1993) cites a network design application where, in order to protect the network from link failure, several edges-disjoint cycles must be determined. This author also mentions a scheduling application of the 2-PSP where each job must be processed twice by the same machine but technological constraints prevent the repetition of identical job sequences.

De Kort (1993) shows that the 2-PSP is NP-hard by transforming an instance of the Hamiltonian Path problem into a 2-PSP. A similar reasoning can be applied to the case where $m > 2$. While the undirected TSP is also an NP-hard problem, medium size instances can easily be solved in practice (see, e.g., Padberg and Grötschel, 1985; Applegate, Bixby, Chvátal and Cook, 2003). A natural question is then whether TSP algorithms can be used to provide good or optimal m-PSP solutions. A partial answer is obtained by applying the following "Krarup heuristic" (Krarup, 1975): solve a first TSP on G (exactly or by means of a heuristic) and remove from the graph all edges of the TSP solution; repeat until m Hamiltonian cycles have been obtained. Krarup shows that even if the TSPs are solved optimally, this heuristic does not always yield an optimal m-PSP solution.

In addition to Krarup's seminal contribution, only a few scientific articles are available on the m-PSP. De Kort (1991) derives a lower bound based on the solution of a capacitated transportation problem and develops a branch-and-bound algorithm (De Kort, 1993) based on a 3-index formulation. Numerical results for $n \leq 60$ and $m = 2$ are reported. Finally, De Kort and Volgenant (1994) have analyzed a generalization of the 2-PSP in which each cycle contains each vertex at most once and a penalty is incurred for vertices not included in a cycle.

Our aim is to provide exact solution procedures for the undirected m-PSP with a general value of m. For this we develop several algorithms using an integer linear programming formulation and a relaxation of the problem. The first algorithm is a branch-and-cut algorithm based on the 3-index formulation of De Kort. The other algorithm uses a 2-index relaxation which give possible solutions thanks to a branch-and-cut algorithm. The resulting graph is then decomposed into triconnected components and the branch-and-cut based on the 3-index formulation is applied on each component. This offers the advantage of splitting the initial graph into subproblems of smallest sizes.

2 3-index formulation

The undirected m-PSP can easily be modeled by means of a 3-index formulation. Let $x_{ijk}\,(i < j)$ be a binary variable equal to 1 if and only if edge (i, j) appears on cycle k. The model is then

$$\text{(3-index)} \qquad \text{minimize} \sum_{k=1}^{m} \sum_{i<j} c_{ij} x_{ijk} \tag{1}$$

subject to

$$\sum_{i<h} x_{ihk} + \sum_{j>h} x_{hjk} = 2 \qquad (h \in V;\;\; k = 1, \ldots, m) \tag{2}$$

$$\sum_{\substack{i,j \in S \\ i<j}} x_{ijk} \leq |S| - 1 \qquad (S \subset V,\;\; 3 \leq |S| \leq \lfloor n/2 \rfloor;\;\; k = 1, \ldots, m) \tag{3}$$

$$\sum_{k=1}^{m} x_{ijk} \leq 1 \qquad (i, j \in V;\;\; i < j) \tag{4}$$

$$x_{ijk} = 0 \text{ or } 1 \qquad (i, j \in V;\;\; i < j;\;\; k = 1, \ldots, m). \tag{5}$$

This model can be strengthened through the introduction of any valid inequality for the TSP. To solve this model, we have developed a branch-and-cut algorithm in which we have used the two most useful families of TSP valid inequalities (Grötschel and Padberg, 1985): 2-matching inequalities and comb inequalities.

3 2-index relaxation

A 2-index relaxation can be derived from 3-index by aggregating the decision variables, i.e., by defining binary variables $x_{ij} = \sum_{k=1}^{m} x_{ijk}$. Thus we obtain the model

$$\text{(2-index)} \qquad \text{minimize} \sum_{i<j} c_{ij}x_{ij} \tag{6}$$

subject to

$$\sum_{i<h} x_{ih} + \sum_{j>h} x_{hj} = 2m \qquad (h \in V) \tag{7}$$

$$\sum_{\substack{i,j \in S \\ i<j}} x_{ij} \leq m(|S|-1) \qquad (S \subset V,\ 2m \leq |S| \leq \lfloor n/2 \rfloor) \tag{8}$$

$$x_{ij} = 0 \text{ or } 1 \qquad (i,j \in V;\ i<j). \tag{9}$$

This 2-index model can be used to compute a lower bound on the m-PSP solution value and, as will be shown, it can serve as a basis for an exact algorithm. In addition, it can be reinforced through the inclusion of somes valid inequalities.

Proposition 1 *The 2-matching inequalities*

$$\sum_{(i,j)\in E(H)} x_{ij} + \sum_{(i,j)\in E'} x_{ij} \leq m|H| + (|E'|-1)/2, \tag{10}$$

where $H \subset V$, $E' \subset E$, $|E'| \geq 3$ and odd, $|\{i,j\} \cap H| = 1$ for all $(i,j) \in E'$, and $\{i,j\} \cap \{h,\ell\} = \emptyset$ for all (i,j), $(h,\ell) \in E'$, $(i,j) \neq (h,\ell)$, are valid for 2-index.

Proposition 2 *If m is odd, the comb inequalities*

$$\sum_{(i,j)\in E(H)} x_{ij} + \sum_{\ell=1}^{r} \sum_{(i,j)\in E(T_\ell)} x_{ij} \leq m|H| + m\sum_{\ell=1}^{r}(|T_\ell|-1) - (mr+1)/2, \tag{11}$$

where $H, T_1, \ldots, T_r \subset V$, $r \geq 3$ and odd, $H \cap T_\ell \neq \emptyset$ and $T_\ell \setminus H \neq \emptyset$ $\quad (\ell = 1, \ldots, r)$, and $T_h \cap T_\ell = \emptyset \quad (h, \ell = 1, \ldots, r,\ h \neq \ell)$, are valid for 2-index.

We can also use the projection of valid inequalities for the 3-index model.

Proposition 3 *The projected 2-matching inequalities*

$$\sum_{(i,j)\in E(H)} x_{ij} + \sum_{(i,j)\in E'} x_{ij} \leq m|H| + m(|E'|-1)/2, \tag{12}$$

where $H \subset V$, $E' \subset E$, $|E'| \geq 3$ and odd, $|\{i,j\} \cap H| = 1$ for all $(i,j) \in E'$, and $\{i,j\} \cap \{h,\ell\} = \emptyset$ for all (i,j), $(h,\ell) \in E'$, $(i,j) \neq (h,\ell)$, are valid for the m-PSP.

Proposition 4 *The projected comb inequalities*

$$\sum_{(i,j)\in E(H)} x_{ij} + \sum_{\ell=1}^{r} \sum_{(i,j)\in E(T_\ell)} x_{ij} \leq m|H| + m\sum_{\ell=1}^{r}(|T_\ell| - 1) - m(r+1)/2, \tag{13}$$

where H, $T_1, \ldots, T_r \subset V$, $r \geq 3$ and odd, $H \cap T_\ell \neq \emptyset$ and $T_\ell \setminus H \neq \emptyset$ $(\ell = 1, \ldots, r)$, and $T_h \cap T_\ell = \emptyset$ $(h, \ell = 1, \ldots, r,\ h \neq \ell)$, are valid for the m-PSP.

The role played by this two projected constraints is different: the projected 2-matching inequalities are dominated by the valid 2-matching inequalities but the projected comb inegalities are better than the valid comb inegalities since they reinforce the relaxation.

To solve our problem, we have developed a branch-an-cut algorithm which solve this relaxation with constraints (10) and (13). As soon as a solution is found, we must check that the solution is a m-PSP feasible solution. To do this, we split the resulting graph into triconnected components and we verify that these subgraphs can be decomposed into m Hamiltonian paths using the 3-index algorithm. If all these subgraphs can be decomposed, at least one m-PSP feasible solution is obtained; otherwise, we add constraints, which eliminate the infeasible subgraph into the 2-index relaxation and the branch-and-cut process continues.

4 Computational results

The algorithms just described were coded in C++ and run on a Compaq AlphaServer DS20 biprocessor EV6/500. All integer linear programs were solved with CPLEX 6.6. TSPs were solved using the Padberg and Rinaldi (1991) code. The algorithms were stopped after 1800 CPU seconds. Tests were performed on TSPLIB instances for $m = 2$ and 3 to facilitate future comparisons between different algorithms. Computational results are summarized in Table 1. The table headings are as follows:

n: number of vertices in G;

m: number of edge-disjoint cycles in the solution;

OPT: optimal m-PSP solution value;

SECONDS: CPU time in seconds (The symbol - indicates that the algorithm did not found the optimal solution within 1800 seconds).

Table 1 show that the 2-index algorithm clearly outperforms than the 3-index algorithm. This can be explained by the two followings observations. First, the Krarup heuristic gives a good upper bound. Second, the 2-index relaxation provides a quite sharp lower bound. We also do computational experiments on randomly generated euclidean and non euclidean instances and for $m = 4, 5$. These additional experiments confirm our observations on the TSPLIB instances. We observed that the complexity increase strongly with m and n. In our computational tests, the larger instance we solved within 1800 seconds is an instance with $n = 144$ and $m = 2$.

Table 1: Success rate and computation time for the two algorithms on TSPLIB instances.

				3-index Algorithm	2-index Algorithm
Problem	n	m	OPT	SECONDS	SECONDS
burma14	14	2	7537	1	0
burma14	14	3	12902	24	0
gr17	17	2	4915	3	0
gr17	17	3	9005	91	0
gr21	21	2	6900	11	0
gr21	21	3	12486	295	0
gr24	24	2	3147	130	0
gr24	24	3	5614	0	0
fri26	26	2	2218	39	0
fri26	26	3	3974	-	0
bayg29	29	2	3737	26	0
bayg29	29	3	6554	-	1
bays29	29	2	4694	12	0
bays29	29	3	8332	-	1
eil51	51	2	982	-	7
eil51	51	3	1737	-	14
eil76	76	2	1257	-	11
eil76	76	3	2141	-	87
rat99	99	2	2928	-	465
rat99	99	3	5128	-	577

5 Conclusion

We have presented new valid inequalities and algorithms for the undirected *m-Peripatetic Traveling Salesman Problem*, a variant of the classical *Traveling Salesman Problem*. To our knowledge, this is the first time exact results are presented for $m \geq 3$.

References

Applegate, D., Bixby, R., Chvátal, V., and Cook, W., 2003, "Implementing the Dantzig-Fulkerson-Johnson Algorithm for Large Traveling Salesman Problems", *Mathematical Programming, Series B* 97, pp 91–153.

De Kort, J.B.J.M., 1991, "Lower Bounds for Symmetric K-Peripatetic Salesman Problems", *Optimization* 22, pp 113–122.

De Kort, J.B.J.M., 1993, "A Branch and Bound Algorithm for Symmetric 2-Peripatetic Salesman Problems", *European Journal of Operational Research* 70, pp 229–243.

De Kort J.B.J.M., and Volgenant, A., 1994, "On the Generalized 2-Peripatetic Salesman Problem", *European Journal of Operational Research* 73, pp 175–180.

Grötschel, M., and Padberg, M.W., 1985, "Polyhedral Theory", in *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys, eds, Wiley, Chichester, pp 251–305.

Krarup, J., 1975, "The Peripatetic Salesman and Some Related Unsolved Problems", *Combinatorial Programming, Methods and Applications*, B. Roy, ed, Reidel, Dordrecht, pp 173–178.

Padberg, M.W., and Grötschel, M., 1985, "Polyhedral Computations", in *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys, eds, Wiley, Chichester, pp 307–360.

Padberg, M.W., and Rinaldi, G., 1991, "A Branch-and-Cut Algorithm for the Resolution of Large-Scale Symmetric Travelling Salesman Problems", *SIAM Review* 33, pp 60-100.

Wolfter Calvo, R., and Cordone, R., 2003, "A Heuristic Approach to the Overnight Security Service Problem", *Computers & Operations Research* 30, pp 1269–1287.