The Network Design Problem with Elastic Probit Stochastic User Equilibrium

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1 Introduction

Traditionally, a deterministic (Wardrop) User Equilibrium (UE) condition has been adopted as the proxy of users' responses to changes in the network. The two main drawbacks of assuming UE are: (i) it is unrealistic to assume users have perfect information, (ii) undesirable characteristics arise when UE is embedded in a NDP including (a) changes of the active path set that make the feasible region non-convex, (b) non-uniqueness of path flows that prevent formulation of the NDP as a single level optimisation program (via the sensitivity analysis of the UE flows) and (c) existence of multiple optima. We argue in this paper that, despite the computational burden of the SUE model, it overcomes most of these obstacles.

There has been some research looking at the NDP with an SUE model (see Davis, 1994). However, most of the work has adopted a Logit SUE model that assumes Weibull-distributed random error terms. This model is well known to possess poor realism for network problems, particularly in the way it neglects path overlaps. An improvement is to adopt a nested Logit SUE, but this cannot be applied to arbitrary network structures. There remains one long-established model that has been known for many years to deal with such problems, namely the Probit SUE, and this will be the approach adopted in the present paper.

This paper aims to formulate and develop an algorithm to solve the NDP with SUE (with an extension to an elastic demand case). In the next section we formally define the condition for Probit users' equilibrium and extend it to a case with elastic demand. The mathematical formulation of the NDP with SUE is then presented in Section 3. An initial attempt to analyse the property of the NDP with the Probit SUE model is conducted in Section 4. Section 5 presents the sensitivity analysis expression of SUE through its fixed point formulation. A by-product of this derivation is an expression for the Jacobian of the fixed point formulation, leading to the development of an optimization algorithm discussed in the last section.

2 Probit Users' Equilibrium with Elastic Demand

The network itself is represented by a directed graph consisting of N nodes, with $\mathcal{A}$ the set of connecting links. The demand matrix, $\boldsymbol{q}$, has entries, q_{rs}, representing the travel demand from origin r to destination s, where $r, s = 1 \ldots N$. The vector of link flows is $\mathbf{x}$, with link costs $\mathbf{t}(\mathbf{x})$, so that $t_a(x_a)$ is the cost of travelling along link $a \in \mathcal{A}$ when the link flow is x_a. The set of paths connecting node r to node s is K_{rs}. The (binary) link-path incidence matrix, Δ^{rs}, with elements $\delta_{a,k}^{rs}$, denotes the links comprising each path connecting node r to s. An assignment of flows to all paths is denoted by the vector $\mathbf{f}$, with $f_k^{rs} \geq 0 \ \forall k, r, s$. The assignment $\mathbf{f}$ is *feasible* for demand $\mathbf{q}$ if and only if

$$\sum_{k \in \mathcal{K}_{rs}} f_k^{rs} = q_{rs} \quad \forall r, s, \tag{1}$$

and the (convex) set of feasible path flows is denoted F. With $\mathbf{c}(\mathbf{f})$ the vector of path costs, the cost of the k-th path is

$$c_k^{rs}(\mathbf{f}) = \sum_{a \in A} t_a(\mathbf{x}(\mathbf{f}))\, \delta_{a,k}^{rs}.$$

In an attempt to incorporate into this traffic assignment model the fact that individual drivers have their own assessment of both network conditions and of the cost of taking different routes (including their personal preferences for some routes over others), their route choice behaviour can be assumed to follow *a random utility model*. This assumption leads to a different set of equilibrium flows: the Stochastic User Equilibrium (SUE). Let the perceived cost of the k-th route be (the random variable) C_k^{rs}, then

$$C_k^{rs} = c_k^{rs} + \varepsilon_k$$

where $c_k^{rs} = c_k^{rs}(\mathbf{f})$ is the mean perceived route cost and the random errors $(\varepsilon_1, \varepsilon_2, \ldots)$ follow some joint probability density function with zero mean vector. Many error structures have been proposed for SUE, not only the commonly used independent Weibull and multivariate Normal that lead to the Logit and Probit models respectively, but also more general cross-nested logit models, mixed error component models, and gamma link component distributions. In this paper we assume that the random errors follow the multivariate Normal distribution (i.e. Probit model).

We then define the proportion, $P_k^{rs}(\mathbf{c})$, of drivers travelling from r to s choosing the k-th path as those who perceive that this is the cheapest route, given path costs $\mathbf{c}$;

$$P_k^{rs} = \Pr\left(C_k^{rs} \le C_j^{rs}\ \forall j \in K_{rs}, j \ne k\right) = \Pr\left(\varepsilon_k^{rs} + c_k^{rs} \le \varepsilon_j^{rs} + c_j^{rs}, \forall j \in K_{rs}, j \ne k\right), \quad (3)$$

where Pr(.) denotes probability.

For most choices of error distribution, in particular for the multivariate normal distribution that leads to the Probit case, calculation of this probability (for every path) is one of the central computational difficulties in determining the SUE flows. One of the main attractions of using independent Weibull-distributed error terms (the Logit model) is that these probabilities are straightforward to evaluate exactly. For most other distributions the path choice probabilities cannot be written in closed form and instead must be approximated analytically or estimated numerically. This is indeed the case for our Probit SUE model. The Stochastic User Equilibrium (SUE) is defined to be a path flow vector such that:

$$f_k^{rs} = q_{rs} P_k^{rs}(\mathbf{c}(\mathbf{f})) \quad \forall k \in K_{rs}, \forall r, s. \quad (4)$$

A network path flow vector satisfying SUE will be denoted **f***.

It is unrealistic to assume that no matter how severe congestion becomes, no one will decide to postpone (or cancel) their intended journey. Typically, this decision has been encapsulated by a demand function describing the relationship between the number of travellers and the cost of their desired journey. In extending the SUE formulation to the elastic demand case, one way to deal with this (see Maher *et al.* 1999) is to use the expected minimum perceived route cost as the argument of the demand function. This is perhaps the nearest equivalent quantity to the experienced (minimum) route cost that occurs in the UE case. The problem with this formulation is that, in deciding whether or not to travel, drivers are described as comparing the cost (disutility) of not travelling with the perceived minimum route cost *averaged across the entire population*. It seems more plausible that each driver compares the cost of not travelling with the minimum route cost *as judged by them alone*, i.e. compares the option of not travelling with the single best route as they see it, regardless of the thoughts of the rest of the population (that are unknown to them). This individual-based approach to the variable demand SUE problem derives entirely from random utility theory and leads to a rather simple statement of the variable demand SUE flows.

Consider the drivers' decision as a choice between the utilities of different routes, measured against the disutility of not travelling (or going later or by a different mode that is outside of the road network). For each OD pair, the option of *"no travel"* can be represented in the network by a pseudo-link that provides drivers with another choice of OD route. The perceived cost of no travel will vary across the population of drivers and so this cost has a random part associated with it, just like the costs of travelling along the other links in the network. However, since the option of not travelling does not suffer from congestion, it is assigned a constant cost (the mean disutility across the population). Writing the cost of not travelling as $C_0 = c_0 + \varepsilon_0$, the proportion of drivers not travelling is:

$$P_0^{rs} = \Pr\left(c_0^{rs} + \varepsilon_0^{rs} \le c_k^{rs} + \varepsilon_k^{rs} \,\forall k \in K_{rs}\right) = \Pr\left(c_0^{rs} + \varepsilon_0^{rs} \le \min_{k \in K_{rs}}\left\{c_k^{rs} + \varepsilon_k^{rs}\right\}\right)$$

exactly like the route choice probabilities in (3). The option of whether or not to travel sits alongside the choice of which route to take, each having a distribution of perceived costs across the population of drivers. The variable demand SUE formulation can then be written as an extended version of the fixed point definition of SUE

$$f_k^{rs} = q_{rs} P_k^{rs}(\mathbf{c}(\mathbf{f})) \quad \forall k \in K_{rs}^0, \forall r, s, \tag{5}$$

where $K_{rs}^0 = K_{rs} \cup \{0\}$, with f_0^{rs} the number of drivers electing to not travel, and q_{rs} is now the (fixed) total number of potential drivers, some of whom choose route 'zero' and do not travel.

3 Problem Formulation of NDP with Elastic Probit Users' Equilibrium

Network Design refers to the *'improvement'* of a network achieved by changing the network design parameters. It is usual that the network is evaluated when it has reached (some sort of) equilibrium; in this paper we assume the equilibrium state follows the Elastic Probit model as expressed in (5). The NDP with the imposed equilibrium constraint naturally presents itself as a mathematical program with equilibrium constraints (MPEC). The mathematical program is an optimisation problem, adjusting the network design parameters (link capacities, tolls etc) in order to maximise the network's performance as measured by our objective function, and this is constrained to be evaluated with the network flows at SUE. With objective function $g(.)$, and network design parameters $\mathbf{s}$, the NDP is

$$\max g(\mathbf{f}, \mathbf{s}) \quad \text{such that} \quad \mathbf{f} = \mathbf{f}^*(\mathbf{s}) \tag{6}$$

The NDP can present a non-smooth objective function 'surface' with multiple optima in a high dimensional space. Sophisticated numerical methods are required to tackle such problems.
There are two possible strategies for solving the NDP. First is to reduce the problem to a single level optimization problem via a *sensitivity expression* of the equilibrium condition. The second strategy is to apply an existing or tailor-made optimization algorithm to the problem as expressed in (6). In applying an existing optimization algorithm to (6), we need information regarding the Jacobian of the fixed point condition, despite the involvement of the Monte Carlo simulation. In Section 5, we calculate a sensitivity analysis expression for the fixed point condition of SUE and propose a method to calculate the Jacobian of the SUE fixed point condition (based on Daganzo,

1979). First, in the next section we present an initial investigation of the property of the Probit SUE and the NDP in (6).

4 Investigation of the NDP with SUE

Consider the two link network with total demand $Q = 11$, with a single network design parameter representing a toll, τ, imposed on Link 1, and link cost functions

$$C_0(x_0) = c_0 + \varepsilon_0 \qquad \text{with } \varepsilon_0 \sim N(0, \sigma_0^2)$$

$$C_1(x_1) = 10 + \tau + x_1 + \varepsilon_1 \qquad \text{with } \varepsilon_1 \sim N(0, \sigma_1^2)$$

$$C_2(x_2) = 60 + x_2^2 + \varepsilon_2 \qquad \text{with } \varepsilon_2 \sim N(0, \sigma_2^2)$$

As described in section 4.4, the demand variation is modelled by a 'pseudo-link' with cost C_0, and x_0 denoting the number of drivers not travelling. We explore the behaviour of this network, that is to say the SUE link flows, as demand (via c_0) and toll are varied. Different values for the variance of the choice probabilities are calculated, including the limiting UE case. Figure 1 depicts the UE and SUE flows on the network for various settings of c_0 and τ. As expected, as the cost of staying at home, c_0, increases, more people are forced onto the network. Note that the transition of flow between the links as the network parameters change occurs smoothly for SUE. This may be contrasted with UE flows that change between routes abruptly (following the complementarity nature of the UE condition). This can be seen as the *'sharp corners'* on the surfaces of the UE flows in Figure 1 and the cross sections of these surfaces in Figure 2.

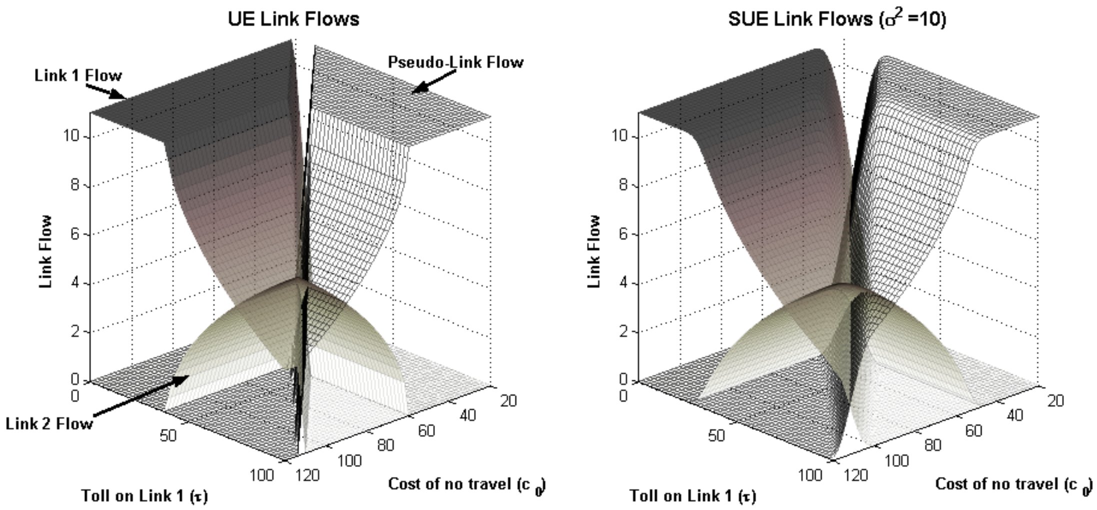

Figure 1: Variable Demand UE and SUE flows on the two-link network.

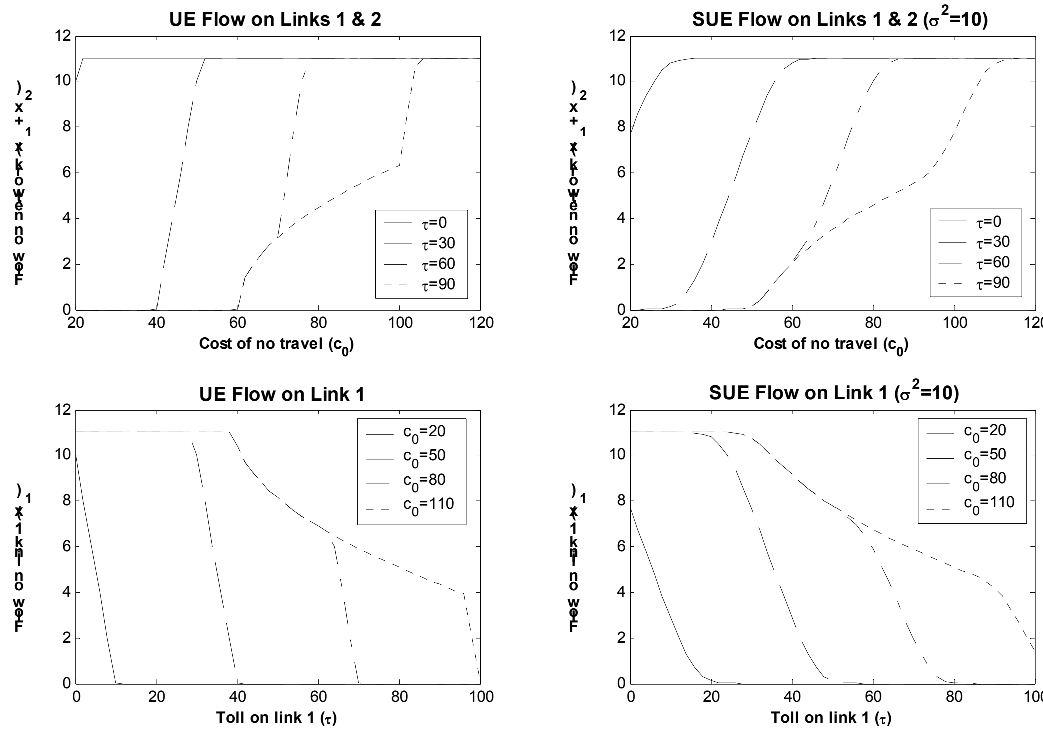

Figure 2: The UE and SUE flows on the two-link network as demand and toll are changed.

Consider the UE and SUE flows as the toll on link 1 varies. The toll determines the relative attractiveness of link 1 or link 2, as can be seen in Figure 2 by the shape of the curves when τ =60, 90; they demonstrate the change of set of used paths in UE case (from one to two paths or links in this case). The same behaviour occurs for the SUE case *but the transitions in flow occur smoothly*. In addition to examining the link flows and their dependence on the underlying parameters of the network, other quantities can also be measured. For example, the revenue generated by the toll on link1 is simply $R = x_1\tau$. From Figure 3, notice that the revenue surface with UE network flows has two local maxima, whereas with SUE flows there is a single global maximum for the revenue surface.

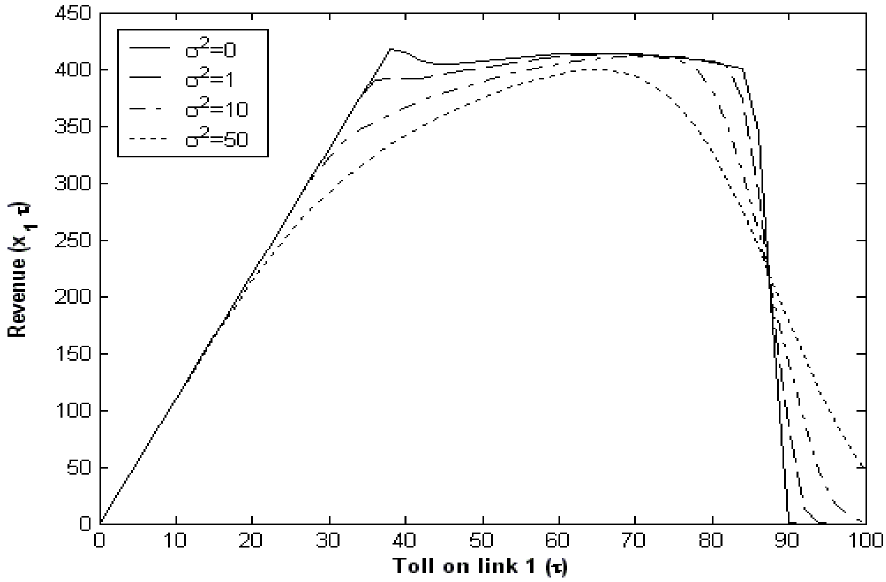

Figure 3: Revenue at fixed demand for UE and SUE with several variances

5 Sensitivity Analysis Expression of Probit Users' Equilibrium

Calculating the SUE flows at any given setting of the network design parameters is time consuming. A local linear approximation to the (surface of) SUE flows can be obtained via sensitivity analysis and this is sufficient to understand the changes in the SUE flows that result from small changes to the network design parameters. For SUE with *fixed* demand, Clark & Watling (2002) derived an expression for the leading order linear changes in the link flows resulting from perturbations of the network design parameters (link cost function or OD demand changes). The SUE path flows can be written as a function of the design parameters, **s**, that can include link cost parameters, OD demands and covariance matrix terms:

$$\mathbf{f}^* = \mathbf{f}^*(\mathbf{s})$$

with **f*** a solution to the variable demand SUE fixed point problem (5). The vector of path cost functions is then $\mathbf{c}(\mathbf{f},\mathbf{s})$. Consider the function

$$\mathbf{h}(\mathbf{f},\mathbf{s}) = \mathbf{f} - \mathbf{P}(\mathbf{c}(\mathbf{f},\mathbf{s}))\mathbf{q}\,.$$

For any given 'setting', $\mathbf{s_0}$, of the design parameters, $\mathbf{h}(\mathbf{f}^*,\mathbf{s}_0) = \mathbf{0}$ since $\mathbf{f}^* = \mathbf{f}^*(\mathbf{s}_0)$ are the corresponding SUE link flows. The Clark & Watling (2002) result is that the leading order approximation to the SUE flows at a new 'setting' of the design parameters, **s,** is given by

$$\mathbf{f}^*(\mathbf{s}) \cong \mathbf{f}^*(\mathbf{s}_0) - \mathbf{J}_1^{-1}\mathbf{J}_2(\mathbf{s}-\mathbf{s}_0)\,, \tag{7}$$

where the Jacobian matrices $(\mathbf{J}_1,\mathbf{J}_2)$ are the derivatives of $\mathbf{h}(\mathbf{f}(\mathbf{s}_0),\mathbf{s}_0)$ with respect to **f** and **s** respectively. Following section 4, this result for fixed demand SUE extends immediately to the case of variable demand SUE by extending the path set to include a constant cost pseudo-link for each OD pair. The main element of $\mathbf{J}_1$ and $\mathbf{J}_2$ is the path choice Jacobian, comprising derivatives of the path choice probabilities with respect to path costs. The formulation and numerical method to calculate $\mathbf{J}_1$ and $\mathbf{J}_2$ are also proposed in Clark and Watling (2002), following Daganzo (1979).

6 Optimization Algorithms for Solving NDP

From the previous section, we have developed an approach to calculate the Jacobian of the fixed point condition, $\mathbf{h}(\mathbf{f}(\mathbf{s}),\mathbf{s})$. Note that this approach already takes account of the stochastic nature of the random error terms. Also note (see Section 4) that the objective function 'surface' is smooth since the set of active paths does not change for SUE, unlike the UE case.

With this expression for the Jacobian of the fixed point condition, gradient-based optimization algorithms can be used to solve the NDP with SUE, as it is stated in (6). Alternatively, one could exploit the sensitivity expression, (7), that gives approximate SUE flows as the design parameters change, by replacing the fixed point condition in (6) with this sensitivity expression. The sensitivity expression does not require Monte Carlo simulation to calculate the new equilibrium flows as the design variables change. Any appropriate optimization algorithm can also be applied to this problem.

These two contrasting approaches have different merits and drawbacks. By using the fixed point formulation as it is in the NDP, concerns about the accuracy of the approximation of the path flows at the final equilibrium point are minimised. On the other hand, the algorithm may suffer from the computational burden of calculating the Jacobian of the fixed point condition at every iteration. Using the sensitivity analysis expression in the NDP resolves this computational problem, since the algorithm does not need to conduct any MC simulation during the optimization process. However, the accuracy of the approximation of the equilibrium flows may be questioned.

In the presentation we will investigate and compare these two approaches to solving the NDP with SUE, discussing some of the theoretical issues arising from employing SUE rather than UE in the NDP, and reporting numerical results for both small and realistic sized networks.

References

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