Rethinking the Wait Model at Transit Stops

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1 Introduction

In an urban transit system where the service is perceived in terms of frequency of the different lines, the mathematical description of the route choice strategy is not trivial, because the wait at the stop, as pointed out in many studies in the last 30 years, is a complex phenomenon which requires a specific analysis (e.g. Chriqui and Robillard [1975], Marguier [1981], Gendreau [1984], Spiess [1984]). In order to develop an effective model for planning the service, it is very important to define a representation of the wait at transit stops which is at once consistent with users' behaviour, mathematically sound and practically usable within the more general assignment models.
This is the reason that has induced us to rethink globally to the *wait problem*, by analyzing the stochastic process of vehicle and passenger arrivals at transit stops and the relations among them, without assuming as given facts some hypotheses that are usually adopted. Indeed, some works written during the 80's [Marguier, 1981; Gendreau, 1984; Marguier and Ceder, 1984] already pointed out that some modelling choices are not easily justifiable, although often utilized in the following years.

† Our colleague Stefano Pallottino sadly passed away on April the 11^{th}.

2 Modelling assumptions

In order to address the wait problem at a transit stop served by one or many lines, we have to introduce several simplifying hypotheses relative to three distinct spheres: the relations among vehicle arrivals of different lines and passenger arrivals, the relations among vehicle arrivals of the same line, the users' knowledge of the network and of its performances. In the following we present briefly the assumptions on which we will base our mathematical framework, referring to Gentile, Nguyen and Pallottino [2003a, 2003b] and Billi [2003] for further details about the modelling choices made and their consequences.

With reference to the first sphere, we assume that the vehicle arrivals of different lines at the stop are statistically independent, and that the same is true for the passenger arrivals with respect to vehicle arrivals.

With reference to the second sphere, the analysis of the stochastic process of the vehicle arrivals of a same line at a given stop induces to introduce some assumptions that, however, are perfectly justifiable with respect to the physical phenomenon. In Billi [2003], the hypotheses made on the headway probability distribution between two successive vehicles are made explicit (among these, also the possible absence of a probability density function). The form that seems to be more appropriate to represent the headway probability density function for a given line ℓ_i is the Gamma function with positive integer parameter m_i (also called *Erlang* function), which takes into account the degree of service regularity [Larson and Odoni, 1981]. From this is derived the probability density function of the *waiting time* denoted by $f_i(w)$:

$$f_i(w) = \begin{cases} \lambda_i \, \mathrm{e}^{-m_i \lambda_i w} \displaystyle\sum_{k=0}^{m_i - 1} \frac{(m_i \lambda_i w)^k}{k!}, & \text{if } w \geq 0; \\ 0, & \text{otherwise;} \end{cases}$$

where λ_i is the *frequency* of line ℓ_i, that is the inverse of the headway expected value.
A physical explanation of parameter m_i associates the passage time of a given vehicle with the passage times of the m_i previous vehicles: after m_i consecutive passages there is a loss of memory within the arrival process. If we set $m_i = 1$, the Erlang function reduces to the *exponential* distribution (total absence of memory within the arrival process); the opposite case is found when m_i tends to plus infinity, obtaining the *uniform* distribution (regular vehicle passages however long is the arrival sequence).

With reference to the third sphere, in the following we will consider users that perceive the sequence of vehicle arrivals of each line only in terms of its frequency and level of regularity, and that know relatively well the network topology and the transit stop locations. Moreover, we will assume that each user waiting at a stop is able to evaluate with sufficient accuracy, for each line ℓ_i, serving that stop, the expected value s_i of the *travel time* to reach its destination once boarded the line, which we will utilize as a simplified measure of generalized cost.

Finally, we assume that each user is able to board any vehicle that he chooses to ride, suffering in case the cost of *discomfort* as a function of the on-board passenger density [Nguyen and Pallottino, 1988].

3 Behavioral hypotheses

Each user is a rational decision maker doing his choices with the aim of minimizing his own generalized cost. In a static model such as that we are referring to, the choice of a user is made *before the beginning of the trip* and implies the iterated choice sequence of the stop to reach on foot where to wait, of the lines to board and, for each of them, of the stop where to alight. The classical representation of this kind of choice is a *hyperpath* connecting the origin of the trip to its destination and having the diversion points in the stop nodes through the waiting hyperarc [Nguyen and Pallottino, 1998]. Key elements of the hyperpath are the weights associated with the branches of each waiting hyperarc, that represent the probability of boarding a vehicle of the different lines of that specific line set, conditional on having reached that stop.

Let us consider then a single transit stop, a set L of lines serving it, and a user that during his trip is there waiting to board one or more lines belonging to L in order to reach his destination. The probability of boarding line $\ell_i \in L$ measures the chance that a vehicle of such line is the first one arriving at the stop which is perceived as *attractive* by the user during his waiting process, in the sense that it is convenient to board that vehicle instead of keep waiting.

Note that, differently from what it is usually assumed in the literature, in this paper we do to conceive in principle the *attractive set* as a collection of lines that are considered always attractive by the user during the entire waiting process. Indeed, in general we assume that, before starting his trip, the user has already defined the lines of L he wishes to board and, for each of them, the period of his waiting process where he considers the line to be attractive. In other words, we assume that the attractive set of a user is defined as a function of the time t elapsed form his arrival at the stop waiting without success; for this reason we will call it *dynamic set* and denote it by $D(t)$, defined for $0 \le t \le u_D$, where u_D is the maximum waiting time, i.e. the first instant when with certainty at least one vehicle has arrived at the stop while considered attractive (eventually, $u_D = +\infty$).

In the behavioural model we take into consideration the time spent by the user waiting at the stop without any attractive arrival; this time is an information that the user acquires during the wait and that influences the perception of the expected remaining waiting time of each line. For example, let us assume that line ℓ_i is perfectly regular ($f_i(w)$ is the uniform distribution function) and has headway $u_i = 20$ min and line time $s_i = 30$ min. When the user reaches the stop and starts waiting, the expected value of the travel time is 10 + 30 = 40 min. After 10 min waiting, the expected value of the remaining waiting time is 5 min and the remaining travel time has decreased to 5 + 30 = 35 min (instead, the total travel time has increased, because to the 35 min we have to add the 10 min already waited).

4 Properties of the dynamic set

Let $D(t)$, $0 \leq t \leq u_D$, be the dynamic set which has been defined "a priori" by the user. A *remaining time* function $RT_D(t)$, $0 \leq t \leq u_D$, is associated with $D(t)$, expressing for each instant $\tau \in [0, u_D]$ the time $RT_D(\tau)$ that remains to wait and to ride in order to reach the destination, given that no attractive arrival has occurred during the interval $[0, \tau)$. The condition defining the attractivity of a line ℓ_i at time τ is given by:

$$s_i \leq RT_D(\tau)\,; \tag{1}$$

in this case, in fact, it is more convenient to board an arriving carrier of the line than to keep waiting at the stop, while in the opposite case it is not convenient to board. We say that $D(t)$ is *instantaneously attractive* at time τ if it is constituted by all lines $\ell_i \in L$ that satisfy condition (1); moreover $D(t)$ is *globally attractive* if and only if it is instantaneously attractive for each $\tau \in [0, u_D]$.

In [Gentile, Nguyen and Pallottino, 2003b] it is shown that, assuming the monotonicity of the remaining waiting time of each line (that is, when the time spent waiting in vain at the stop increases, the remaining waiting time of each line does not increase), then the remaining time function of a globally attractive dynamic set is monotonically not increasing. From this property of monotonicity derives that, if a line $\ell_i \in L$ is attractive, it is such for a time interval $[0, t_i]$, with $0 < t_i \leq u_D$ (a line which is not attractive at time $\tau = 0$ is never attractive). Moreover, the globally attractive dynamic set exists, is unique and varies (reducing itself) a finite number of times. It is defined, by construction, as:

$$D(\tau) = \{\ell_i \in L : s_i \leq RT_D(\tau)\}, \quad \forall 0 \leq \tau \leq u_D\,. \tag{2}$$

Without loss of generality, let us consider that the lines are ordered based on their travel times and that these are different from each other, that is $s_1 < s_2 < \ldots < s_n$, with $n = |L|$; moreover, we will denote by $L_j = \{\ell_1, \ell_2, \ldots, \ell_j\}$ the set of the j faster lines in terms of travel times. Consider the time t_j when $RT_D(t_j) = s_j$. Because $RT_D(t)$ is a monotonically non increasing function, from (2) we have that $D(t_j) = L_j$ and that there exists a time $t_{j+1} < t_j$ at which $RT_D(t_{j+1}) = s_{j+1}$ (or $t_{j+1} = 0$ and $RT_D(0) \le s_{j+1}$). In the time interval $(t_{j+1}, t_j]$ spent waiting (or $[0, t_j]$ if $t_{j+1} = 0$) the dynamic set does not change (we say that it is "static"). Moreover, the dynamic of $D(t)$ is described at the most by n time intervals where $D(t)$ is static. In general, we denote by L_q the set of lines that are attractive during the whole definition interval $[0, u_D]$ and by L_r, $r \le q$, the set of lines that are attractive at time $\tau = 0$; thus the lines $\ell_{r+1}, \ldots, \ell_n$ are never attractive.

5 Travel time expected value and line probabilities

In [Gentile, Nguyen and Pallottino, 2003b] is developed the theoretical framework required to formulate the function $RT_D(t)$ and the probabilities to board the different lines. Here we provide only the most important elements; let u_i denote the upper bound of the definition interval relative to the function $f_i(w)$ for line ℓ_i ($+\infty$ in the case of unbounded distribution); while $F_i(w)$ is the cumulative distribution function of the waiting time. In particular, we are interested in its complement:

$$\bar{F}_i(w) = 1 - F_i(w) = \int_w^{u_i} f_i(t)\, dt\,.$$

The probability density function of the waiting time of line ℓ_i, conditional to having already waited in vain until time τ is:

$$f_{i|\tau}(w) = \begin{cases} \dfrac{f_i(w)}{\bar{F}_i(\tau)}, & \text{if } w \in [\tau, u_i]; \\ 0, & \text{otherwise;} \end{cases}$$

the complement of the corresponding cumulative distribution function is:

$$\bar{F}_{i|\tau}(w) = \int_w^{u_i} f_{i|\tau}(t)\, dt = \begin{cases} \dfrac{\bar{F}_i(w)}{\bar{F}_i(\tau)}, & \text{if } w > \tau; \\ 1, & \text{otherwise.} \end{cases}$$

Recalling the meaning of the instants t_j for the different attractive lines, let us consider the case when the user has waited in vain until time τ, with $t_{j+1} < \tau \le t_j$, when the lines $\ell_1, \ldots, \ell_j$ are attractive, while the lines $\ell_{j+1}, \ldots, \ell_r$ are not anymore attractive. If first attractive arrival occurs at time w , the (density of) probability $\gamma_i^{\,j}(w|\tau)$ that it is relative to a vehicle of line ℓ_i, with $i \le j$, is given by:

$$\gamma_i^j(w\,|\,\tau) = f_{i|\tau}(w) \prod_{k=1,k\neq i}^{j} \bar{F}_{k|\tau}(w) = \frac{f_i(w)}{\bar{F}_i(w)} \prod_{k=1}^{j} \frac{\bar{F}_k(w)}{\bar{F}_k(\tau)}.$$

On this basis, the share of travel time expected value relative to the case where the first attractive arrival occurs within the time interval $[\tau, t_j]$, is given by:

$$\Psi^j(\tau, t_j) = \int_{\tau}^{t_j} \left(\sum_{i=1}^{j} \left(w + s_i - \tau\right) \frac{f_i(w)}{\bar{F}_i(w)} \right) \prod_{k=1}^{j} \frac{\bar{F}_k(w)}{\bar{F}_k(\tau)} dw. \tag{3}$$

Knowing that $RT_D(t_j) = s_j$, by means of the latter formula it is possible to provide a "local" description of the remaining time function $RT_D^{\,j}(t)$ relative to the generic interval $(t_{j+1}, t_j]$ where the static set of attractive lines is L_j:

$$RT_D^j(\tau) = \Psi^j(\tau, t_j) + s_j \prod_{k=1}^{j} \frac{\bar{F}_k(t_j)}{\bar{F}_k(\tau)}, \quad j = q, \ldots, r. \tag{4}$$

In the above formula it is to be intended $t_q = u_D$; moreover, in the case of unbounded headways, we have in general $q = 1$, except for the case of exponential distribution where the classical results are valid and replace the above formulation.

Equation (4) expresses the remaining travel time as the sum of its share $\Psi^j(\tau, t_j)$, relative to the case where the first attractive arrival occurs within the time interval $[\tau, t_j]$, and the complementary share s_j, opportunely weighted by the probability that the user has to wait in vain until time t_j.
Using (4) it is possible to verify if line ℓ_{j+1} is attractive for at least a part of the waiting process. In fact, it is sufficient to check that $RT_D^{\,j}(0) \le s_{j+1}$. In this case, line ℓ_{j+1} is never attractive, that is $r = j$ and $t_{j+1} = 0$. Otherwise we determine the time t_{j+1} such that $RT_D^{\,j}(t_{j+1}) = s_{j+1}$, and then we have $RT_D(\tau) = RT_D^{\,j}(\tau)$, for each $\tau \in (t_{j+1}, t_j]$. The expected travel time is given by:
$ET = RT_D(0)$;
while the probability of utilizing the vehicles of line $\ell_i \in L_r$ is given by:

$$\pi_i = \sum_{h=\max\{i,q\}}^{r} \prod_{j=h+1}^{r} \bar{F}_j(t_j) \int_{t_{j+1}}^{t_j} \frac{f_i(w)}{\bar{F}_i(w)} \prod_{j=1}^{h} \bar{F}_j(w)\, dw. \tag{5}$$

Clearly $\pi_i = 0$ for the non attractive lines ℓ_i, i $= r + 1, \ldots, n$.

6 Computational aspects

The local form of $RT_D^{\,j}(t)$ suggests a "backward construction" of the remaining time function. The determination of q can be achieved starting from $q = 1$ and updating q with j+1 whenever during the backward construction of the remaining time function we find $t_{j+1} \ge u_{j+1}$.

The above formulation requires in general the computation of many integrals, which might be somewhat time consuming unless introducing approximations whose propagation is difficult to control. In the case of Erlang functions this can be avoided, noting that all the functions to be integrated are a sum of different terms whose generic form is:

$$\int \beta \, \mathrm{e}^{-\alpha x} x^{v} dx = -\beta \, \mathrm{e}^{-\alpha x} \sum_{k=0}^{v} \frac{v!}{k!} \frac{x^{k}}{\alpha^{v-k+1}} + c \, ;$$

where α and β are real parameters, while v is a nonnegative integer parameter and c in an additive constant. On this basis it is possible to determine the expected value of the travel time and the line probabilities through computations that are easy to organize in order to reduce the number of operations and to control the propagation of approximation errors. This approach is inspired to the one proposed by Gendreau [1984].

7 Necessity of a dynamic model

Consider the following example with only two lines whose waiting time has a uniform distribution: $u_1 = u_2 = 50$ min, $s_1 = 30$ min and $s_2 = 50$ min. Applying the proposed model we obtain the following globally attractive dynamic set: $D(t) = L_2$ for $0 \le t \le 10$ and $D(t) = L_1$ for $10 < t \le 50$, with an expected value of the travel time $RT_D(0) = 54.53$ and line probabilities $\pi_1 = 0.82$, $\pi_2 = 0.18$.

Applying instead the classic static approach, which is valid only for exponential distribution, we obtain L_1 as optimal attractive set with a travel time expected value of 55 (the travel time expected value of the set L_2 is 56.67). However, in this case at the beginning of the wait the user renounces to board the vehicles of the second line, although this would be convenient in terms of travel time; that is, the assumption of passenger's rational behaviour fails.

8 Conclusions

We have described a new, well founded, model for the line choice at the stops of an urban transit network, where it is possible to take into account the level of service regularity for each line and for each one of its segments. We have also suggested a computational technique that guaranties the efficiency of the model and the possibility of plugging it into any transit assignment model based on hyperpaths.

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