Leveraging Traffic Flow Modeling and Game Theory for Pricing and Logistical Systems

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1 Introduction

Dynamic pricing has become very important in recent years due to its wide range applicability in a variety of industries. The rapid development of information technology, the Internet and E-commerce as well as the success of the Direct-to-Customer (DTC) business model, have had a strong influence on the development of dynamic pricing, as they provide a company with the flexibility of dynamically changing the prices of its products. Dynamic pricing has been extensively studied by researchers from a variety of fields. The paper by McGill and van Ryzin [6], and the references therein, provide a thorough review of revenue management and pricing models. Elmaghraby and Keskinocak [4] review the literature and current practices in dynamic pricing in industries where capacity or inventory is perishable and fixed in the short run. Yano and Gilbert [8] review models for joint pricing and production under a monopolistic setup.

In this paper, we consider a make-to-stock fluid model. Make-to-stock is the standard for a large number of industries such as retail. In addition, Avram, Bertsimas and Ricard [1] and Bertsimas and Paschalidis [3] show that fluid models provide good production and inventory policies in a variety of settings. However, the models considered in these papers do not address pricing. The model we propose and study in this paper addresses joint pricing and inventory management. We borrow ideas, modeling techniques, and algorithms from the transportation area.

A key and novel characteristic of the model in this paper is that instead of considering a traditional demand model that assumes an a priori relationship between price and demand with fixed parameters, we consider a model that relies on how price and level of inventory affect the time a unit of product remains in inventory. We refer to this time spent in inventory as delay or sojourn time. The impetus of considering a delay model is motivated from: (1) The widespread recording - particularly in information technology enabled industries - in data warehouses of entrance times and exit times of products in inventory systems, which makes this delay data available. (2) The delay data being internal and easily extractable from data warehouses, as opposed to demand data, which is external, and therefore not controlled by the company. (3) In an environment where price does not vary a lot with time, the estimation of the relationship between price and demand, which is used as an input to the pricing models in the literature, can be quite inaccurate. However, because of the moderate to high variability of inventories with time, the estimation of the relationship between inventory level and sojourn time can be more accurate. A few companies such as *Amazon.com* are currently using sojourn time information to control their inventories and adjust their pricing policies. Furthermore, we believe that the new and fast growing technology of Radio Frequency IDentification chips (RFIDs) -that is projected to dominate the manufacturing and retail sectors within the next five years- will provide a strong motivation and a large number of applications for this type of models.

In this paper, we consider (i) a multi-product and dynamic environment, (ii) a dynamic production capacity shared amongst all products, and (iii) the presence of competition. In particular, the goals of this paper are the following:

- We borrow the Dynamic Network Loading Model from traffic modeling to introduce a general model of dynamic pricing and inventory management in logistical and supply chain systems.
- The model is a general continuous-time fluid dynamics model that jointly considers production, inventory and pricing decisions. It considers a price inventory relationship, where the pricing parameters are not fixed but rather are an output of the model.
- We establish that the delay approach we propose in this paper directly connects with the traditional demand approach.
- In order to study the efficient solution of the general model, we consider a discretized version of the model and illustrate its efficient solution.
- We incorporate competition, analyze the best response model solved by each competing retailer and establish sufficient conditions for the existence of a Nash Equilibrium. We further propose an iterative relaxation algorithm that allows us to compute an equilibrium policy.

2 Notation and Formulation of the Dynamic Pricing Model

We consider a multi-product inventory system that we represent by a directed network with two nodes O and D, and n links joining these two nodes. Node O represents the arrival of a product to the warehouse and node D represents the delivery of this product to the customer.

Each link joining O and D corresponds to a distinct product that the company is selling and is indexed by i, $i \in \{1, ..., n\}$.

We first assume that the company under study is a Stackelberg leader, and as a result is a price setter. Therefore, competitors' prices are functions of the price of the company under study. These functions can be estimated in practice using regression on the competitors' prices and the prices of the company under study. Subsequently, in the paper, we consider a more general competitive setting where several retailers compete, each optimizing their profits that depend on competitors' policies (i.e., retailers solve a Best Response Problem simultaneously). We establish existence of Nash Equilibrium production and pricing policies for the competing retailers.

Note that this model is similar to the Dynamic Network Loading fluid model studied in the context of transportation (see for example, Bernstein and Friesz [2] and the references therein, Wu *et al.* [7]). This model is often referred to as the DNL Model.

Inputs of the Dynamic Pricing Model

Link variables:
$CFR(t) =$ Shared production capacity rate at time t. $p_i^c(p_i(.)) = (p_{i,j}^c(p_i(.)), j \in \{1, ..., J(i)\})$, vector of price functions of companies competing on product i. $D_i(I_i) = T_i(I_i, p_i, p_i^c)$: product sojourn time function, that is the total time a newly produced unit of product i spends in the inventory system, given an inventory I_i, a unit price $p_i(I_i)$, and a set of competitors' price functions $p_i^c(.)$. $c_i(t)$: production cost of product i at time t. $h_i(t)$: inventory cost of product i at time t.

Time variables: $[0, T]$: production period.

Outputs of the Dynamic Pricing Model

Link variables:
$U_i(t)$: cumulative production flow of product i during interval $[0, t]$ $u_i(t)$: production flow rate of product i at time t. $V_i(t)$: cumulative sales flow of product i during interval $[0, t]$; $v_i(t)$: sales flow rate of product i at time t; $I_i(t)$: inventory (number of units of product) i at time t; $p_i(I_i(t))$: sales price of one unit of product i given an inventory $I_i(t)$; $s_i(t)$: exit time of a production flow of product type i entering at time t $(s_i(t) = t + D_i(I_i(t)))$.

Time variables: $[0, T_\infty]$ is the interval of time from when the first unit of product is produced to the first instant all products have been sold.

The study of the general pricing model does not require any assumption on the functional form of the unit price function $p_i(I_i)$. Instead, the unit price function is an output of the model. However, in this paper, when we analyze a discretized version of the pricing model, we assume that the unit price function $p_i(.)$ is linear as a function of the inventory. Also, notice that the unit price function $p_i(I_i(t))$ depends on time only through the time-dependence of the inventory $I_i(t)$.

We consider a sojourn time function $D_i(I_i(t)) = T_i(I_i(t), p_i(I_i(t)), p_i^c(p_i(I_i(t))))$ that represents the total time it takes to sell, at time t, a newly produced unit of product i, given a level of inventory $I_i(t)$, a unit price $p_i(I_i(t))$ and a set of competitors' prices $p_i^c(p_i(I_i(t)))$. Notice that the product sojourn time function $D_i(I_i(t))$ resembles the time to traverse a link in a transportation network.

Model Formulation

The model in this paper includes the added complexities that it considers a shared production capacity environment, it incorporates the pricing component, and finally, it is placed in the framework of dynamic optimization.

Before formulating the model, we describe the setting and the assumptions. We consider a competitive setting where:
B1) There are multiple products.
B2) The total production capacity rate is bounded by a non-negative capacity flow rate function $CFR(.)$.
B3) There is no substitution between products.
B4) The company under study faces holding costs but no setup costs.
B5) The demand is deterministic.
B6) The unit price $p_i(.)$ depends on the inventory I_i.

The study of the general pricing model does not require any assumption on the functional form of the unit price function $p_i(.)$. Instead, the unit price function is an output of the model. Notice that Assumption B6 allows us to consider a variety of models for the unit price functions. Examples of such models include linear functions of the type $p_i(I_i) = p_i^{max} - \frac{p_i^{max} - p_i^{min}}{C_i} I_i$ as well as nonlinear functions of the type $p_i(I_i) = \frac{p_i^{max}}{(\frac{p_i^{max}}{p_i^{min}} - 1)\frac{I_i}{C_i} + 1}$, where C_i denotes the inventory capacity, p_i^{max} the maximum allowable price, and p_i^{min} the minimum allowable price.

Dynamic Pricing Model:

$$\text{Maximize} \quad \sum_{i=1}^{n} \int_0^{T_\infty} [p_i(I_i(t))v_i(t) \; - \; c_i(t)u_i(t) - h_i(t)I_i(t)]dt \tag{1}$$

$$\text{s.t.} \quad \frac{dI_i(t)}{dt} = u_i(t) \; - \; v_i(t), \quad \forall i \in \{1, ..., n\} \tag{2}$$

$$V_i(t) = \int_{\omega \in W} u_i(\omega) d\omega, \quad \forall i \in \{1, ..., n\}, \quad \text{where} \quad W = \{\omega : s_i(\omega) \le t\} \tag{3}$$

$$U_i(0) = 0, \quad V_i(0) = 0, \quad I_i(0) = 0, \quad \forall i \in \{1, ..., n\}$$

$$\sum_{i=1}^{n} u_i(t) \le CFR(t), \tag{4}$$

$$u_i(.) \ge 0, \quad \forall i \in \{1, ..., n\}, \qquad CFR(.) \ge 0. \tag{5}$$

Remarks:

- The objective of the company is to maximize its profits. That is, by subtracting production costs and inventory costs from sales.

- The link dynamics equations (2) express the change in inventory at time t as the difference between the production and the sales flow rates.

- The flow propagation equations (3) describe the flow progression over time. Note that a production flow entering link i at time t will be sold at time $s_i(t) = t + D_i(I_i(t))$. Therefore, by time t, the cumulative sales flow of link i should be equal to the integral of all production inflow rates which would have entered link i at some earlier time ω and would have been sold by time t.

- Furthermore, if the product exit time functions $s_i(.)$ are continuous and satisfy the strict First-In-First-Out (FIFO) property, then the flow propagation equations (3) can be equivalently rewritten as

$$V_i(t) = \int_0^{s_i^{-1}(t)} u_i(\omega)d\omega, \qquad \forall i \in \{1, ..., n\}. \tag{6}$$

  Notice that $s_i^{-1}(t)$ is the time at which a unit of product i needs to be produced so that it is sold at time t. Furthermore, under the strict FIFO condition, a unit of product i, entering the queue at time t, will be sold only after the units of product i, that entered the queue before it, are all sold. In mathematical terms, this is equivalent to the product exit time functions $s_i(.)$ being strictly increasing. As a result, defining the production time $s_i^{-1}(t)$ makes sense.

- In general, the DPM Model is a continuous-time non-linear optimization problem. The non-linearity of the model comes from the unit price as a function of the inventory, as well as the integral equation (3). In this formulation, the known variables are the product sojourn time functions $D_i(.)$, the production and inventory costs $c_i(.)$ and $h_i(.)$, and the total capacity flow rate function $CFR(.)$. The unknown variables we wish to determine are $u_i(t)$, $v_i(t)$, $U_i(t)$, $V_i(t)$ and $I_i(t)$. Notice that integral equation (3), which connects the production to the sales schedules through the delays incurred in the system due to price and inventory, makes this problem hard to solve.

In Kachani and Perakis [5], we investigate when the FIFO property holds. We examine conditions on the product sojourn time functions $D_i(.)$ and on the production flow rates $u_i(.)$. We also establish the existence result below. This result illustrates that under general assumptions, the DPM Model possesses an optimal solution.

Theorem 1 *[5] Assume that the following conditions hold:*
(E1) The price inventory functions $p_i(I_i)$ are continuously differentiable and bounded from above by scalars p_i^{max}.
(E2) The product sojourn time functions $D_i(.)$ are continuously differentiable, and there exist two non-negative constants B_{1i} and B_{2i} such that for every inventory level I_i, $0 \leq B_{1i} \leq D_i'(I_i) < B_{2i}$.

(E3) The shared capacity flow rate function $CFR(.)$ *is Lebesgue integrable, non-negative and does not exceed* $\min_{1,\ldots,n} \frac{1}{B_{2i}-B_{1i}}$.
Then, the Dynamic Pricing Model has an optimal solution.

Remark: This theorem suggests that the maximum variation of the delay in terms of the inventory connects with the total production rate for all products. As a result, when the shared capacity (which is an upper bound on the total production rate) is small (or large) and the maximum variation of the delay in terms of the inventory is large (or small) then the Dynamic Pricing Model has a solution.

In this paper, we study a discretized version of the model which gives rise to a quadratic optimization problem. We illustrate how we can determine both the optimal production levels and the optimal pricing policies. We present an efficient iterative relaxation algorithm to not only solve this problem but also solve the more general problem that incorporates competitors' behavior. This iterative relaxation algorithm utilizes equilibrium techniques from traffic assignment and fictitious play from game theory to compute a Nash Equilibrium production and pricing policies.

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