An Expression of Long Term Demand Uncertainty for the Robust Traffic Equilibrium Problem

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1 Introduction

There has been significant amount of work in the last fifty years in the area of static traffic assignment with deterministic assumptions. However, it is well known that both the long-term and short term demands in a transportation network are uncertain and can at best be estimated within certain accuracy. Most of the efforts in the recent past have concentrated their efforts on studying reliability in terms of the uncertainty in link capacities (Lo and Tung, (2003), Chen et al. (2002)). Lo and Tung (2003) propose a model to study relatively minor network disruptions in within day traffic operations under the assumption that the capacity is distributed as a uniform distribution. They use Mellin's transforms to solve the problem which may not yield closed form results in the case of a general distribution where non-central moments are required. An equally important, if not more, in the planning process is to estimate the impact of the long term demand uncertainty. This problem is more difficult because there is no direct relation between the demand and the total system travel time in the convex formulation proposed by Beckmann (1952). Most of the existing literature is mainly focused on studying the interaction between the travel demand and supply on transportation networks in the short-term. Here, short-term demand is defined in the context of studying day to day/hour-to-hour (real time) fluctuations in demand against the long-term demand in which the temporal demand variations occur over a period, say months/years. The fundamental question that this research addresses is; *what would be the impact on the network design decisions by taking into account the long-term uncertainty in demand?* A first step to answer this question would be to address the following the question: *Can we get closed form analytical solutions for the mean and variance of travel time when the long term demand is uncertain in the static traffic assignment problem?*

The study of uncertainty in transportation network planning is emerging as an important topic of recent research. This extended abstract is concerned with accounting for long-term uncertainties in forecasted demand value. As explained before, the long-term demand uncertainty differs from short-term demand uncertainties mainly in the time-frame, where the traffic equilibrium is assumed to be achieved some time into the future with the variation in the demand. Long-term demand uncertainty plays a crucial role in network design investment decisions, where the optimal investment decisions are to be made in the present, accounting for the *demand fluctuations* into the future. This research solves the traffic assignment problem when the long term demand is an uncertain quantity. A closed form expression for the system's expected value and variance are obtained. We believe that designing networks to minimize this variance would result in robust solutions. Further, the equilibrium properties under such kind of long term demand uncertainty are studied and verified using well known transportation test networks.

2 Motivation

The basic UO and SO formulations for the static traffic assignment problem are presented below before proceeding with further analysis of the problem.

UO Formulation

$$\min \ \mathbf{Z}(x) = \sum_{\forall a} \int_0^{x_a} t_a(\omega) d\omega$$

s.t.

$$\sum_{\forall k} f_k^{rs} = q_{\tilde{rs}}$$

$$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs}$$

$$f_k^{rs} \geq 0$$

SO Formulation

$$\min \ \mathbf{Z}(x) = \sum_{\forall a} x_a t_a(x_a)$$

s.t

$$\sum_{\forall k} f_k^{rs} = q_{\tilde{rs}}$$

$$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs}$$

$$f_k^{rs} \geq 0$$

Note that the demand is represented as $q_{\tilde{rs}}$ to distinguish that this is a random variable into the long-term decision making process.
A very simplistic way of approaching the problem as it done in practice is to substitute the long term forecasted demand with its expected value. However, by Jensen's Inequality (1906), it is well known that when a function $\Gamma(q_{\tilde{rs}})$ is convex, and then $\Gamma(E(q_{\tilde{rs}}))$ underestimates the true system performance which is $E(\Gamma(q_{\tilde{rs}}))$. Thus, we need a methodology which can explicitly calculate $E(\Gamma(q_{\tilde{rs}}))$. The following research is an attempt in that direction.

3 Formulation and Derivation

Derivation of the path and link travel time random variable distributions

CASE I: The demand $q_{\tilde{rs}}$ is assumed to be normally distributed with a mean *m*, and variance d^2, i.e. $q_{\tilde{rs}} \sim N(m, d^2)$
In this research, we also assume that the link performance function is given by:

$t_i(\omega_i) = t_i[1+\alpha(\frac{\omega_i}{C_i})^\beta]$, where t_i is link *i*'s free flow travel time, C_i is the capacity of link *i* which is assumed to be deterministic, α and β are deterministic link specific parameters. This is similar to the BPR function.
Now, for the UO case the objective function reduces to

$$\sum_{\forall a} \int_0^{x_a} t_a(\omega) d\omega = \sum_{\forall a} \int_0^{x_a} t_0[1+\alpha(\frac{\omega_a}{C_a})^\beta] d\omega_a \qquad (1)$$

The integrated expression above is referred as $\mathbf{Z}(x) = \sum_{\forall a} t_0 \left\{ x_a + \frac{\alpha}{(\beta+1)C_a^{\beta}} [x_a^{\beta+1}] \right\}$ (2)

The above equation represents the objective of the UO with the costs given by the BPR type link performance functions.

To find E [Z(x)], the question remains as to how to find the relation between the random OD demand and Z(x). As the capacity of each link is deterministic, the only variable affected by the change in long term demand would be long term link flows on each of the link. It is quite intuitive that if the long term projected demand is more then the corresponding link flows on all the used paths would correspondingly increase as compared to the base case. However, the important question would be

From the Convolution Theorem, it is well known that the sum of independent normal random variables is also a normal random variable. In the formulation if the demand is assumed to be a random normal variable, then one solution (although not unique) is that the path flows are also random normal variables with some arbitrary mean and variance. Further, with the same logic it can be shown that the link flows, x_a are also random normal variables. This will be further proved with experimentation on different networks. This will be clearly demonstrated in the full paper which is a working paper at the University of Texas at Austin.

The objective is to find the noise (volatility) and the expected value of the objective function because of the stochasticity in the long term demand. Assume that the link flows are distributed with mean m_i and standard deviation d_i for all i in a.

$$E[\mathbf{Z}(x)] = E[\sum_{\forall i} t_0 \left\{ x_i + \frac{\alpha}{(\beta+1)C_i^{\beta}} [x_i^{\beta+1}] \right\}]$$

$$\Rightarrow \quad t_0 E[\sum_{\forall i} x_i] + \frac{t_0 \alpha}{(\beta+1)C_i^{\beta}} E[\sum_{\forall i} (x_i^{\beta+1})]$$

$$\Rightarrow \quad t_0 \sum_{\forall i} m_i + \frac{t_0 \alpha}{(\beta+1)C_i^{\beta}} E[\sum_{\forall i} (x_i^{\beta+1})]$$

Now, the question is what is the value of $E[\sum_{\forall i} (x_i^{\beta+1})]$?

First, we need to calculate

$$E[x_a^{\beta+1}] = \int_0^{\infty} (x_a)^{\beta+1} \rho(x_a) dx_a$$

This is calculated first by moment about the mean (the central moments of the normal distribution).

$$E[(x_a - m)^{\beta+1}] = \int_{-\infty}^{\infty} (x_a - m)^{\beta+1} \frac{1}{\sqrt{2\pi d^2}} \exp\{- {(x_a - m)^2}/{2d^2}\} dx_a$$

Substituting $\frac{(x_a - m)}{d} = z$

$$\Rightarrow \frac{d^{\beta+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z)^{\beta+1} \exp\{-\frac{(z)^2}{2}\} dz$$

When $\beta+1$ = odd number, then the integral is a product of an odd function and an even function. So, the integration is symmetric and is zero.

When $\beta+1$ = even number, say *2n*, then,

$$E[(x_i - m_i)^{\beta+1}] = \frac{d_i^{\beta+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z)^{\beta+1} \exp\{-\frac{(z)^2}{2}\} dz$$

$$= \frac{d_i^{\beta+1}}{\sqrt{2\pi}} \sqrt{\pi} \frac{1.3.......\beta}{2^{\frac{\beta+1}{2}}} \{\frac{1}{2}\}^{-\frac{(\beta+2)}{2}}$$

$$\Rightarrow [1.3.......\beta] d_i^{\beta+1} \qquad (3)$$

$$\text{Hence, } E\,[\mathbf{Z}(x)] = \begin{cases} t_0 \sum_{\forall i} m_i + \frac{t_0 \alpha}{(\beta+1) C_i^{\beta}} \sum_{\forall i} [1.3.......\beta] d_i^{\beta+1}, & \text{if } \beta+1 = even \\ t_0 \sum_{\forall i} m_i & \text{if } \beta+1 = odd \end{cases}$$

We can also calculate the volatility (variance) of the objective function.

$$Var[\mathbf{Z}(x)] = t_0^2 \sum_{\forall i} Var[x_i] + \frac{\alpha^2}{(\beta+1)^2 (C_i)^{2\beta}} \sum_{\forall i} Var[x_i^{\beta+1}]$$

$$Var[x_i^{\beta+1}] = E[x_i^{2(\beta+1)}] - E^2[x_i^{\beta+1}]$$

$$E[x_i^{2(\beta+1)}] = \frac{d_i^{2(\beta+1)}}{\sqrt{2\pi}} \sqrt{\pi} \frac{1.3........(2\beta+1)}{2^{\beta+1}} (\frac{1}{2})^{-\frac{(2\beta+3)}{2}}$$

As, $(2\beta+2)$ is even, this is equal to $[1.3.5.....(2\beta+1)] d_i^{2(\beta+1)}$

$$E^2[x_i^{\beta+1}] = \begin{cases} [1.3.5.....\beta]^2 d_i^{2(\beta+1)}, & when \beta+1 = even \\ 0, & when \beta+1 = odd \end{cases}$$

$$\text{Hence } Var[x_i^{\beta+1}] = \left\{ [1.3.5.....(2\beta+1)] d_i^{2(\beta+1)} + \begin{cases} [1.3.5.....\beta]^2 d_i^{2(\beta+1)}, & when \beta+1 = even \\ 0, & when \beta+1 = odd \end{cases} \right\}$$

$$= \begin{cases} [(1.3.5.....(2\beta+1)) + (1.3.5.....\beta)^2] d_i^{2(\beta+1)}, & when \beta+1 = even \\ [(1.3.5.....(2\beta+1)] d_i^{2(\beta+1)}, & when \beta+1 = odd \end{cases} \qquad (4)$$

Finally, $Var[\mathbf{Z}(x)] =$

$$t_0^2 \sum_{\forall i \in a} d_i^2 + \frac{\alpha^2}{(\beta+1)^2 (C_i)^{2\beta}} \sum_{\forall i \in a} \begin{Bmatrix} [(1.3.5.....(2\beta+1)) + (1.3.5.....\beta)^2] d_i^{2(\beta+1)}, when \beta+1 = even \\ [(1.3.5.....(2\beta+1)] d_i^{2(\beta+1)}, \qquad when \beta+1 = odd \end{Bmatrix}$$

The above results give the central moments for a standard Gaussian distribution. However, we require the non-central moments of the Gaussian distribution. The general result for moments of a multivariate Gaussian distribution has been a problem of interest in the mathematics community and until recently has not been solved completely (?). A recent result by Triantafyllopoulous (2002) gives a more efficient procedure for calculating the moments of the multivariate Gaussian distribution. We propose an alternative method for the univariate Gaussian distribution which is simpler yet elegant to get a closed form solution for this problem. We convert the non-central moments in terms of the moments of the standard normal variable using Binomial Theorem. The results derived above in (3) and (4) are used for simplification.

$$E[\sum_{\forall i} (x_i^{\beta+1})] = E[\sum_{\forall i} (x_i - m_i + m_i)^{\beta+1}] = \sum_{\forall i} E\left[\sum_{j=0}^{\beta+1} {}^{\beta+1}C_j (x_i - m_i)^j m_i^{\beta+1-j} \right]$$

$$= \sum_{\forall i} \sum_{j=0}^{\beta+1} \binom{\beta+1}{j} m_i^{\beta+1-j} E\left[(x_i - m_i)^j \right] = \sum_{\forall i} \sum_{j=0}^{\beta+1} \binom{\beta+1}{j} m_i^{\beta+1-j} [1.3.......(j-1)] d_i^j \text{ ; where } j \text{ is even.}$$

Illustration: For $\beta = 4$, the value of $E[\sum_{\forall i} (x_i^{\beta+1})]$ is equal to $\sum_{\forall i} \left[m_i^5 + 10 m_i^3 d_i^2 + 15 m_i d_i^4 \right]$. For small values of β the same result can be derived using the Moment Generating Function.

Hence $E\,[\mathbf{Z}(x)] = \sum_{\forall i} \left\{ t_0 m_i + \frac{t_0 \alpha}{(\beta+1) C_i^\beta} \sum_{j=0}^{\beta+1} \binom{\beta+1}{j} m_i^{\beta+1-j} [1.3.......(j-1)] d_i^j \right\}$; where j is even.

For $\beta = 4$, $E\,[\mathbf{Z}(x)] = \sum_{\forall i} \left\{ t_0 m_i + \frac{t_0 \alpha}{(\beta+1) C_i^\beta} [m_i^5 + 10 m_i^3 d_i^2 + 15 m_i d_i^4] \right\}$ (5)

The above expression (5) shows that as the mean and variance of the link flows increase there is a corresponding increase in the expected value of the objective function, which can be thought of as a surrogate of the total system travel time. This result is consistent with the expectation that as there is greater expected travel time on the links, there is a corresponding increase in the average total system travel time.

Variance can be calculated similarly using the equations (4) and (5).

$$E[\mathbf{Z^2}(x)] = \sum_{\forall i} \sum_{j=0}^{2\beta+2} \binom{2\beta+2}{j} m_i^{2\beta+2-j} [1.3.....(j-1)] d_i^j \text{ ; where } j \text{ is even.}$$

Hence

$$Var[\mathbf{Z}(x)] = \sum_{\forall i} \sum_{j=0}^{2\beta+2} \binom{2\beta+2}{j} m_i^{2\beta+2-j} [1.3.....(j-1)] d_i^j - \left\{ \sum_{\forall i} \sum_{j=0}^{\beta+1} \binom{\beta+1}{j} m_i^{\beta+1-j} [1.3.....(j-1)] d_i^j \right\}^2 \quad (6)$$

Equation (6) shows the variability of the objective function with the mean and variance of the link travel times. It is consistent with our expectations that the variability of the whole system would increase if there is a larger variability on each of the links. For example, if the demand is unknown with a larger variance the system has to be designed taking into consideration the increased variance in travel time apart from the increase in expected total system travel time.

4 Conclusions

This research developed analytical closed form solutions when the long term planning demand is stochastic. The relationships between the demand and path flows in the network are obtained by numerical experimentation and the expected value and the variance of the system is obtained in a closed form. A clear advantage of the proposed approach is that the robust solutions for the traffic network assignment and design can be obtained without extensive simulations to capture stochasticity. Numerical results have not been included in the abstract but the correctness of the approach will be demonstrated in the full paper which is still under preparation. This research should form the basis for the thrust of the new area of developing robust models in transportation network modeling.

References

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