Vehicle Routing with Time Windows - Introducing New Valid Inequalities

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1 Introduction

The vehicle routing problem (VRP) involves finding a set of routes, starting and ending at a depot, that together cover a set of customers. Each customer has a given demand, and no vehicle can service more customers than its capacity permits. The objective may be to minimize the total distance travelled or the number of vehicles used, or a combination of these. In this paper, we consider the vehicle routing problem with time windows (VRPTW), which is a generalization of the VRP. A solution to the VRPTW must ensure that the service at any customer starts within a given time interval, called a time window. We assume that the time window is hard, i.e. if the vehicles arrives too early, the vehicle must wait until the time window opens, and it is not allowed to arrive late. In the case of soft time windows these can be violated, but then a penalty is imposed. The VRPTW appears in many real life situations, for example deliveries to supermarkets, bank and postal deliveries, industrial refuse collection, school bus routing, security patrol service, and urban newspaper distribution.

In this paper we will focus on an exact method for the VRPTW. Exact methods for solving the VRPTW dates back to Kolen, Rinnooy Kaan, and Trienekens (1987). They were able to solve quite small problem instances to optimality (up to 15 customers). The method was based on dynamic programming and state space relaxation. In this paper we will concentrate on methods based on column generation. The most successful exact column generation approaches for the VRPTW have been based on constrained shortest path relaxations. Desrochers, Desrosiers, and Solomon (1992) used a column generation (Dantzig-Wolfe decomposition) scheme. Halse (1992) implemented a decomposition based on variable splitting (also known as Lagrangean decomposition). Kohl and Madsen (1997) developed an algorithm exploiting Lagrangean relaxation. Kohl, Desrosiers, Madsen, Solomon, and Soumis (1999), Larsen (1999), and Cook and Rich (1999) implemented a Dantzig-Wolfe based decomposition algorithm. Kallehauge (2000) developed an algorithm based on a combination of Lagrangean relaxation and Dantzig-Wolfe decomposition. In all seven cases, the resulting subproblem was a shortest path problem with time window and capacity constraints. The subproblem solution forms a part of the column generation. Even though the subproblem is NP-hard, a pseudo polynomial algorithm exists for one of its relaxations.

2 The model

The problem is formulated as a mixed integer programming problem like in Desrochers, Desrosiers and Solomon (1992). The model contains two decision variables. A triple indexed zero-one variable indicating if a specific vehicle drives directly from a specified customer to another specified customer. And a double indexed variable indicating the time a specific vehicle starts servicing a specific customer. The constraints are

- each customer is serviced exactly once
- every route originates and ends at the depot
- flow conservation is observed
- the time windows of the customers are observed
- the capacity constraints of the vehicles are observed

For a reasonably sized real instance the model has very large dimension making it impossible to solve the model directly by standard software. However the model is structured in such a way that the only coupling constraints are the constraints ensuring that each customer is serviced exactly once. The remaining constraints are only dealing with one vehicle at a time. This means that it is very tempting to use Lagrangean relaxation (LR) or decomposition (for example Dantzig-Wolfe decomposition) such that the problem can be decomposed into a subproblem (a shortest path problem with resource constraints) for each vehicle and a master problem (finding new multipliers). If the vehicles are identical as in the present case all the subproblems will be identical and therefore it is only necessary to solve one subproblem. The solution of the subproblem will result in generating new columns to be used in the master problem and it will provide us with a lower bound for the optimal value of the objective function. In this paper we will use Lagrangean relaxation where the customer assignment constraints are relaxed.

3 The master problem

The master problem consist of determining new multipliers. This can be done by subgradient optimisation, bundle methods, or trust region methods. According to our experience the convergence of subgradient optimisation is very slow in this problem area. We have chosen to apply a trust region method, which in this case results in solving an LP problem in each master iteration. This approach is very similar to the column stabilization approach for the dual problem.

4 The subproblem

The subproblem is an elementary shortest path problem with time windows and capacity constraints (ESPPTWCC) where elementary means that each customer can only participate at most once in the shortest path. The problem is NP-hard. The usual approach is to change the problem slightly by relaxing some of the constraints by allowing cycles. Even though there is a possibility for negative cycles in the graph the time windows and the capacity constraints prohibits infinite cycling. Of course the above mentioned relaxation will not result in a lower bound as good as the lower bound obtained from ESPPTWCC. A compromise could be to forbid cycles of small length (i.e. with a small number of arcs in the cycle). In our approach we have eliminated cycles containing 2 arcs.

The computational time needed for solving the subproblem is increasing if the customer demand is small compared to the capacity of the vehicle (i.e. many customers on each route), if the time windows are wide, and if there are many negative cycles in the network.

5 Branch and bound and acceleration strategies

The column generation approach does not automatically guarantee integer solutions. Often solutions obtained will be fractional. Therefore a branch and bound framework was established. In our approach the following strategies were used:

- Branching on the number of vehicles
- Branching on flow variables
- Branching on time windows

To speed up the computation the following acceleration strategies have been used in our approach:

- Preprocessing to tighten the time windows
- Stopping in the subproblem before optimality as long a negative marginal cost is found

6 Improving the lower bound

In order to reduce the number of nodes in the branch and bound tree and to speed up he solution process it is recommended to introduce additional constraints, usually in the master problem. In the previous work 2-path cuts have been applied (Kohl, Desrosiers, Madsen, Solomon and Soumis, 1999) in the root node with some success. In this paper we will introduce the following new constraints both in the root node and the other nodes in the branch and bound tree:

- 2-path cuts in all the branch and bound nodes
- infeasible path elimination constraints
- odd CAT constraints
- lifted cycle constraints
- precedence constraints

Computational experience shows that these additional constraints will reduce the branch and bound tree and speed up the computations considerably.

7 Computational results

In the paper the above mentioned solution approach will be tested on the Solomon test instances and on some of the Homberger instances. The results show a considerable improvement in computational time and a possibility to solve yet unsolved instances.

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