# Wave Tracking Resolution Scheme for Bus Modelling inside the LWR Traffic Flow Model

Vincent Henn* Ludovic Leclercq*

*LICIT (INRETS–ENTPE)
Rue Maurice Audin
69518 Vaulx-en-Velin cedex – France
{vincent.henn,ludovic.leclercq}@entpe.fr

When trying to evaluate the potential benefits of an infrastructure project involving dedicated lines for buses or of a new public transport management policy such as a bus priority at traffic signals, one needs to use a model able to calculate the resulting congestion (in terms of density), some travel times (or mean speeds) for vehicles or for the bus, etc.

Such a model needs to represent the mutual effects between a bus and the surrounding traffic flow. Indeed, a congested flow can prevent a bus to drive at its desired speed and, conversely, a bus can generate some congestion because it reduces locally the capacity of the road when driving slower than the surrounding vehicles.

In the LWR (Lighthill, Whitham [9] and Richards [11]) traffic flow model framework, some researches have been made to extend the homogeneous representation of the traffic flow by taking into account the buses as punctual moving bottleneck. Leclercq *et al.* [8] proposes a review of these different studies and try to present a unique frame to put them together. This paper will focus on the numerical resolution of this LWR moving bottleneck model.

The basic LWR model is classically solved by using the Godunov scheme [6]. This scheme is based on a spatial discretization (grid of cells) of the different links of the network. Some researches have been lead to extend this scheme by taking into account a moving bottleneck [2, 1]. These works come up against the difficulty to deal with a moving singularity inside a fixed grid of cells. We have so decided to explore another way to solve the LWR moving bottleneck model by using the principle of the "wave tracking" method. In fact, this method uses an event discretization which seems to be well adapted to deal with moving bottlenecks.

We will first shortly review the LWR model and its extension in order to represent a bus and its influence on the surrounding traffic flow. Then, we will present the wave tracking method and explain how it can be applied to the bus modelling issue. We will finish by illustrating the method on a small theoretical application.

# 1 The LWR model and its extension for bus modelling

The LWR model considers traffic as a homogeneous and continuous stream, characterized by three variables: the flow $Q(x,t)$, the density $K(x,t)$ and the flow speed $V(x,t)$. The basic model equations are

- The conservation equation: $\dfrac{\partial K(x,t)}{\partial t} + \dfrac{\partial Q(x,t)}{\partial x} = 0.$
- The flow definition: $Q(x,t) = K(x,t).V(x,t)$.
- An equilibrium (fundamental) relation $Q_E$: $Q(x,t) = Q_E(K(x,t))$, which represents all the equilibrium situations traffic could encounter depending on the road configuration.

Those equations can be synthesized into the following non-linear, hyperbolic conservation equation:

$$\frac{\partial K(x,t)}{\partial t} + \frac{\partial Q_E(K(x,t))}{\partial x} = 0 \tag{1}$$

The basic LWR model only describes the evolution of a homogeneous traffic flow. An extension of this model has been proposed by Newell [10], Lebacque *et al.* [7] and more generally by Leclercq *et al.* [8] to take account of the effect of a moving obstruction corresponding to slower vehicles (buses, lorries...). This extension considers that the traffic states nearby this bottleneck are described by a unique diagram (further called the bottleneck diagram) which does not depend on the speed $V_{\text{bus}}$ of the bottleneck.

With only this hypothesis, it is possible to deduce the traffic states $(K_U^*, Q_U^*)$ upstream, $(K_D^*, Q_D^*)$ downstream and $(K_A^*, Q_A^*)$ nearby the moving bottleneck. In fact, it is possible to demonstrate [7, 8] that the relative flow $q = Q_U - K_U V_{\text{bus}} = Q_D - K_D V_{\text{bus}}$ is conserved on both sides of the moving bottleneck. Furthermore, when the moving bottleneck is active (that is the case where the traffic states are different on both side of the bottleneck due to its presence), the upstream and downstream traffic states are associated with an equilibrium state on the bottleneck diagram where the relative flow is maximal.

The two above properties can be translated graphically by using the fundamental diagrams. The conservation of the relative flow imposes that the traffic states upstream, downstream and nearby the moving bottleneck are linked by a straight line whose slope is $V_{\text{bus}}$, which is called the capacity line (cf. Figure 1). As the relative flow is maximal when the bottleneck is active, the straight line is necessarily tangent to the bottleneck diagram. The tangential point describes the traffic state along the moving bottleneck.

To sum it up, two states are possible:

- If density is between $K_D^*$ and $K_U^*$ the bottleneck is *active*, then the bus constrains the traffic,
- Otherwise, the bus has no effect on the traffic and just drives at its own speed.

It is to remember that the bus can never go faster than the other vehicles, so that if density is too high, its speed is reduced to the traffic speed.

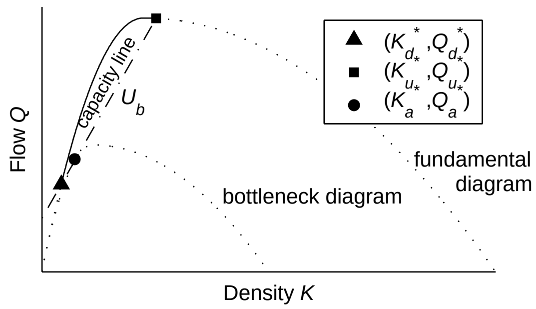


**Figure 1**: The LWR moving bottleneck model

## 2 Application of the wave tracking method to the bus modelling issue

### 2.1 The wave tracking method

The wave tracking method (WT) is a numerical method for the resolution of hyperbolic conservation equations which is based on an explicit propagation of shock waves in the space-time diagram.

Since the speed of shock waves is constant (following the Rankine-Hugoniot formula), they propagate linearly. On the contrary, the frontier of a rarefaction fan is in general not linear and might be quite hard to handle. However, when the flux function is piecewise linear, it appears that the rarefaction fan is composed of constant density zones separated by shock waves.

Thus the idea of the WT method is to approximate the flux function by a piecewise linear function so that there are only shock waves which are easy to propagate. This method has been recently applied in particular to the LWR traffic flow model [3] and can be summarized as followed:

- The fundamental diagram is approximated by a piecewise linear function.
- Density jumps are solved locally and result in the generation of linear shock waves separating constant density zones.
- Each wave propagates linearly till it meets another one (generating a new density jump) or an exogeneous event occurs (change of the color of a traffic signal, beginning of an incident, etc.)
- The propagation is calculated till there is no more wave (and there is only one constant density zone) or till the desired end of the simulation is reached.

The method is proved to converge towards the entropic solution when the approximated fundamental relation tends to the initial one [4].

## 2.2 Application to the bus modelling

The WT method is quite easy to apply to the representation of a bus driving inside a traffic flow. Indeed, its trajectory can be considered as a special wave which is also linear.

Like other vehicles, the bus has a constant speed inside a constant density zone, expect if for example the bus stops to drop or get some passenger. So that, in general, the trajectory of a bus is piecewise linear in the space-time diagram.

Hence, we can consider this trajectory as a wave propagating the "bus" information (instead, for example, of a density jump in the case of a classical shock wave).

When the bottleneck is active and the bus constrains the surrounding traffic flow, its trajectory is a shock wave separating two distinct constant density zones (of density $K_U^*$ and $K_D^*$). Otherwise, it is just an information propagating at the bus speed and the densities upstream and downstream are identical.

In addition to the propagation of waves, we have to consider the following events:

- Some bus related events:
  - intersection of the trajectory of a bus with a shock wave (coming indifferently from upstream or downstream),
  - insertion of a bus into the traffic (exit of a bus lane),
  - vanishing of a bus out of the traffic (entrance into a bus lane),
  - change of speed of the bus (for example at a bus stop or because of an up-hill ramp)
- Other events not related to the bus:
  - intersection of two shock waves
  - exogeneous events (beginning of an incident, change of the color of a traffic signal, etc.)

## 2.3 Resolution of the WT method

An important hypothesis of the WT method is that events can be solved "locally", which implies that they can be solved sequentially, even if they happen at the same time. This property is due to the finite speed of propagation of information (through the different waves) and is still valid when considering a bus inside a traffic flow (cf. [4, p. 37])

Let us denote by $(x, t)$ the position of an event in the space-time diagram.

Solving an event not related to the bus results in the resolution of the following Riemann problem:

$$\mathcal{R}(x,t): \quad \begin{cases} \forall \xi < x, & K(\xi,t) = K_U \\ \forall \xi > x, & K(\xi,t) = K_D \end{cases} \tag{2}$$

Such a problem is classically solved in the WT method by generating a new shock wave (if $K_U > K_D$) or a set of new diverging shock waves (if $K_U < K_D$).

For some bus related event, the following *extended* Riemann problem has to be considered:

$$\mathcal{R}_{\text{bus}}(x,t): \quad \begin{cases} \forall \xi < x, \;\; K(\xi,t) = K_U \\ \forall \xi > x, \;\; K(\xi,t) = K_D \\ x_{\text{bus}}(t) = x \end{cases} \tag{3}$$

It can be shown that such a problem can be solved sequentially by first solving the simple Riemann problem (without bus), and then introducing the bus inside a constant density zone and considering the mutual effects between the bus and the surrounding traffic inside this zone.

The resolution can then be described as follows:

1. Solve the classical Riemann problem $\mathcal{R}(x,t)$. This generates waves originated at $(x,t)$.
2. Identify the constant density zone $Z$ inside which the bus is entering. Denote by $K_Z$ its density and by $V_Z$ the speed of vehicles inside it.
3. Consider the mutual effects between the bus and the traffic:
   - If the traffic is too dense in the zone to allow the bus to drive at its desired speed (that is $V_Z < V_{\text{bus}}$). Then reduce the speed of the bus to the speed of the surrounding flow: $V_{\text{bus}} := V_Z$. Then the wave corresponding to the bus trajectory is not a shock wave and is parallel to the trajectory of any other vehicle.
   - If density of the traffic is such that it can flow around the bus without constraint (that is $K_Z < K_D^*$ or $K > K_U^*$), then the bus propagates at its own speed and there is no density jump (the bus wave is not a shock wave).
   - Otherwise, the bottleneck is active and the traffic is constrained by the bus, two zones are created directly upstream and downstream the position of the bus of respective densities $K_U^*$ and $K_D^*$. This introduction generates two new Riemann problems just upstream the bus $(x^-)$ and downstream $(x^+)$:

     $$\mathcal{R}(x^-,t): \quad \begin{cases} \forall \xi < x^-, \;\; K(\xi,t) = K_Z \\ \forall \xi > x^-, \;\; K(\xi,t) = K_U^* \end{cases} \tag{4}$$

     and

     $$\mathcal{R}(x^+,t): \quad \begin{cases} \forall \xi < x^+, \;\; K(\xi,t) = K_D^* \\ \forall \xi > x^+, \;\; K(\xi,t) = K_Z \end{cases} \tag{5}$$

     Those Riemann problems can result in the generation of one or several shock waves.

One possible flaw of the method could be the generation of a very high number of waves which would mean that it is practically impossible to use. Empirically, an order of magnitude of the number of generated waves has been observed to be close to the product of the number of exogeneous events by the number of segments in the fundamental diagram approximation

## 3 Application

On the contrary to the classical Godunov scheme, the WT method is based on events and not on a fixed discretization grid. Thus it is quite more complicated to implement into a computer program, but not impossible (one can refer to [5] for an example of a detailed implementation).

We have developed an object-oriented program (in $C^{++}$) in order to manage the different events, the resolution of the Riemann problems as well as the generation and propagation of shock waves.

Such a program can represent traffic signals, incidents, capacity reduction (fixed and moving bottlenecks) and can easily be extended in order to incorporate other types of events as soon as they can be described in term of Riemann problems.

We made a simulation on a simple theoretical case in order to represent the possibility of the method. Figure 2 depicts the moving of a bus, stopping and moving off at a traffic signal (NB: in order to isolate the effects of the bus, only one red phase has been represented).

Calculations have been made by considering a 100 point approximation of the fundamental diagram and result in the generation and treatment of less than 200 events and constant density zones. Such a thin approximation makes the space-time diagram almost smooth (whereas it is still piecewise linear).

## 4 Conclusion

This article shows that a numerical scheme based on the wave tracking method is very efficient to solve the LWR moving bottleneck model. In fact, this method uses an event discretization where modifications in the traffic behaviour are propagated by waves. As the moving bottleneck can be considered as a particular wave propagating the "bus information" (jump in density and position when the bottleneck is active or only position in the other case), the numerical resolution of this extension to the LWR model is quite straightforward.

## References

[1] C. F. Daganzo and J. A. Laval. On the numerical treatment of moving bottlenecks. Technical report, Institute of transportation studies, Berkeley, 2003.

[2] F. Giorgi, L. Leclercq, and J.-B. Lesort. A traffic flow model for urban traffic analysis: extensions of the LWR model for urban and environmental applications. In M. Taylor, editor, *Proc. of the 15th International Symposium on Transportation and Traffic Theory*, pages 393–415. Pergamon, 2002.

[3] V. Henn. A wave-based resolution scheme for the hydrodynamic LWR traffic flow model. In *Proc. of the workshop "Traffic and granular flows"*, Delft, The Netherlands, 2003.

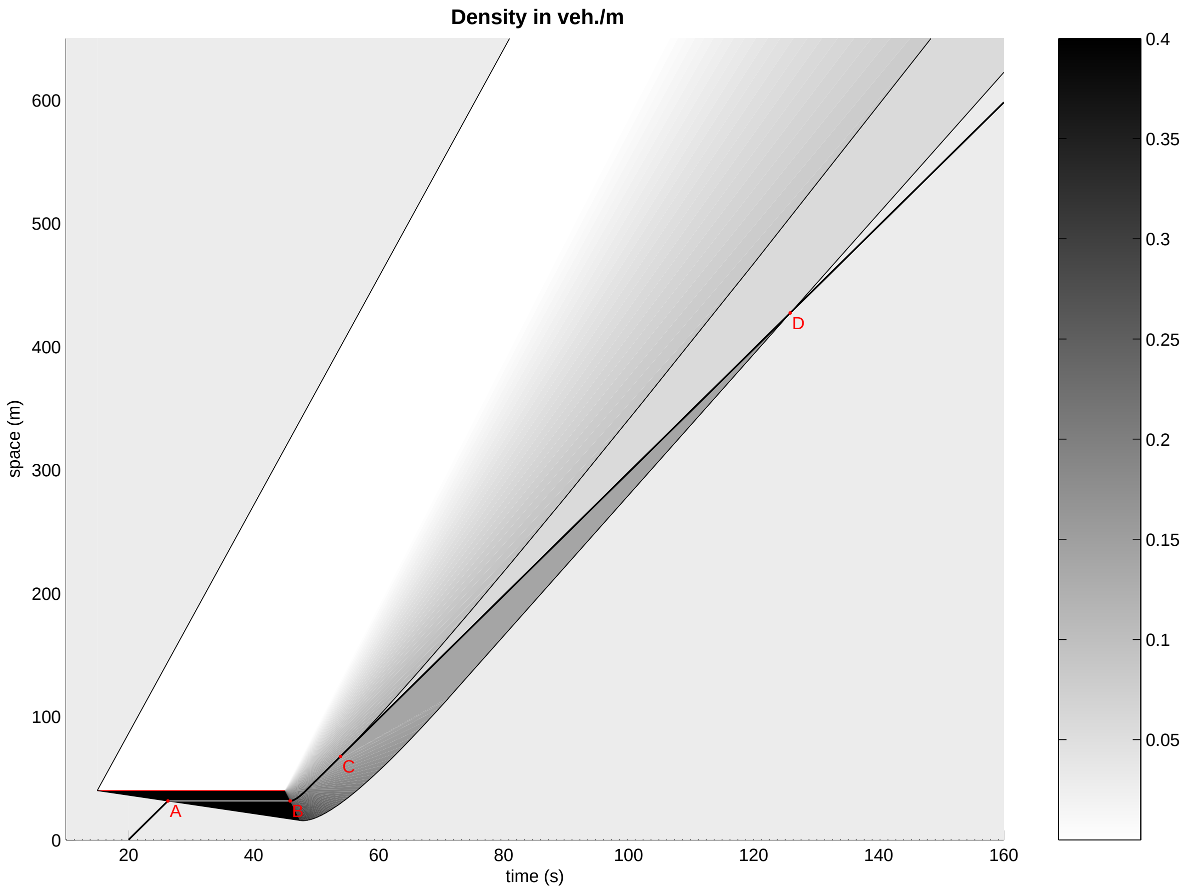


**Figure 2**: Trajectory of a bus inside a traffic flow stopping at a traffic signal. (A) The bus is stopped because of the queue forming behind the traffic signal. (B) The bus can move off, at the speed of the traffic at the beginning and at its desired speed afterwards. (C) The traffic is too dense to pass anymore through the bottleneck: two constant density zones are generated ($K_U^*$ and $K_D^*$). (D) The bus exits the fan due to the traffic signal

[4] H. Holden and N. H. Risebro. *Front tracking for hyperbolic conservation laws*, volume 152 of *Applied mathematical sciences*. Springer, 2002.

[5] J. O. Langseth. On an implementation of a front tracking method for hyperbolic conservation laws. *Advances in engineering software*, 26:45–63, 1996.

[6] J.-P. Lebacque. The Godunov scheme and what it means for first order flow models. In J.-B. Lesort, editor, *Proc. of the 13th International Symposium on the Theory of Traffic Flow and Transportation*. Pergamon, 1996.

[7] J.-P. Lebacque, J.-B. Lesort, and F. Giorgi. Introducing buses in first order traffic flow models. *Transpn. Res. Record*, 1644:70–79, 1998.

[8] L. Leclercq, S. Chanut, and J.-B. Lesort. Moving bottlenecks in the LWR model: a unified theory. In *Proc. of the 83rd Transportation Research Board Annual Meeting*, January 2004.

[9] M. J. Lighthill and G. B. Whitham. On kynematic waves II. A theory of traffic flow on long crowded roads. *Proc. of the Royal Soc.*, A(229):317–345, 1955.

[10] G. F. Newell. A moving bottleneck. *Transpn. Res.-B*, 32(8):531–537, 1998.

[11] P. I. Richards. Shocks waves on the highway. *Oper. res.*, 4:42–51, 1956.