# A Local Search Method for a Pricing Problem on a Transportation Network 

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## 1 Introduction

There is no denying a renewed interest in toll roads, either managed by governments or private societies. Toll roads may help alleviate congestion while putting the monetary burden on the actual users of the infrastructure.

In this presentation we consider the problem of determining a set of optimal tolls on the arcs of a multicommodity transportation network. The problem involves two decision makers acting non cooperatively and in a sequential way. More precisely, we consider the situation where the owner of a private toll highway seeks to maximize revenues raised from tolls set on a subset of arcs of a transportation network, while the commuters aim to travel at minimum cost from their origin to their destination.

This sequential and non cooperative decision-making process can be adequately represented as a bilevel program. It has been introduced by Labbé et al. [4] and applied to the determination of optimal tariffs for a single commodity (respectively multicommodity) transportation problem by Brotcorne et al. [1] and [2]. Recently it has been proved to be an NP Hard problem by Roch et al. [6].

An assumption underlying our model is that congestion is not affected by the rerouting that
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could result from the introduction of tolls. An explicit account of congestion would radically transform the mathematical nature of the model and calls for entirely different algorithmic approaches. Note that model involving congestion issues has for example been recently studied by Chen and Bernstein [3].

In this presentation, we first present the bilevel model for the toll setting problem we consider. Next, we describe a local search algorithm which could be embedded in a metaheuristic procedure. Finally we present some numerical results. Let $G=(\mathcal{N}, \mathcal{A})$ be a transportation network where $\mathcal{N}$ is the node set and the arc set $\mathcal{A}$ is partitioned into the subset $\mathcal{A}_{1}$ of toll arcs and the subset $\mathcal{A}_{2}$ of toll-free arcs. With each arc $a$ of $\mathcal{A}_{1}$ is associated a generalized travel cost composed of a fixed part $c_{a}$ representing the minimal travel cost per unit and an additional unknown toll $T_{a}$, converted to time units. Any $\operatorname{arc} a$ of $\mathcal{A}_{2}$ bears a fixed unit travel cost $d_{a}$.

Let $\mathcal{K}$ denotes the set of commodities. Each commodity $k$ is associated with and origindestination pair $(o(k), d(k))$. The demand vector $b^{k}$ associated with each commodity $k$ is specified by:

$$
b_{i}^{k}= \begin{cases}n^{k} & \text { if } i=o(k) \\ -n^{k} & \text { if } i=d(k) \\ 0 & \text { otherwise }\end{cases}
$$

where $n^{k}$ represents the total number of users of commodity $k$. Finally, $x_{a}^{k}$ denotes the number of users of commodity $k$ on arc $a \in \mathcal{A}_{1}$ and $y_{a}^{k}$ denotes the number of users of commodity $k$ on $\operatorname{arc} a \in \mathcal{A}_{2}$.

Assuming that demand is fixed, users are assigned to shortest paths linking their departure and arrival nodes, for given values of the tolls $T_{a}$ set at the upper level of decision-making. Based on the above notation, the toll setting problem (TSP) can be formulated as a bilevel program with bilinear objectives and linear constraints, where it is understood that the commodity flows $x_{i j}^{k}$ must be part of an optimal solution of the lower linear program parameterized by the upper level toll vector $T$ :

$$
\begin{aligned}
\mathrm{TSP}: & \max _{T, x} \\
& \sum_{(i, j) \in \mathcal{A}_{1}} T_{i j} \sum_{k \in \mathcal{K}} x_{i j}^{k} \\
\min _{x, y} & \sum_{k \in \mathcal{K}}\left(\sum_{(i, j) \in \mathcal{A}_{1}}\left(c_{i j}+T_{i j}\right) x_{i j}^{k}+\sum_{(i, j) \in \mathcal{A}_{2}} d_{i j} y_{i j}^{k}\right) \\
& \text { s.t. } \\
& \sum_{(i, j) \in \mathcal{A}}\left(x_{i j}^{k}+y_{i j}^{k}\right)-\sum_{(j, i) \in \mathcal{A}}\left(x_{j i}^{k}+y_{j i}^{k}\right)=b_{i}^{k} \quad \forall i \in \mathcal{N}, \quad \forall k \in \mathcal{K}, \\
& x_{i j}^{k} \geq 0 \quad \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{1}, \\
& y_{i j}^{k} \geq 0 \quad \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{2} .
\end{aligned}
$$

The leader's objective is to maximize the total revenue which is the sum of the products between toll $T_{a}$ and the number of users on arc $a$. The objective of the follower is to minimize the total cost of the paths selected by the network users. The constraints of the follower's problem are derived from flow conservation (demand) and flow nonnegativity.

In the remainder, we assume that there cannot exists a tariff setting scheme that generates profits and creates a negative costs cycle in the network, and that there exists at least one path composed of free arcs for each origin-destination pair. These assumptions imply that the
lower level optimal solution corresponds to a set of shortest paths and that the upper level profit is bounded from above.
While the leader and the follower act in a noncooperative fashion, we assume that, faced with two equally (un)attractive alternatives, the follower will select the path that yields the highest revenue for the leader, i.e., in all likelihood, the quickest. This assumption is not unrealistic in that, given two equivalent paths, the one generating the highest revenue could be made the most attractive through a minute reduction of one of its tolls.

Finally, we assume that the 'value-of-time' parameter, that allows the conversion from time to money unit, is uniform through the entire population of network users. Our methodology could quite easily been extended to more general situation where users are distributed into classes, each endowed with its own perception of the value of one time unit. In this generalized model, commuters associated with the same origin-destination pair could yet be assigned to different paths.

## 2 Inverse Optimization Program

The interaction between the leader and the follower is twofold. First, for given tolls, the follower solution corresponds to a set of commodity shortest paths.
Next, for each follower solution lying in the inducible region ${ }^{1}$, the best vector $T$ consistent with the lower level solution can be obtained by solving an inverse linear program, formulated as follows. First, let us replace the lower level program by its primal-dual optimality conditions. Denoting by $\lambda$ the dual vector associated with the flow conservation constraints this yields, for fixed arc flows, the single level problem

$$
\begin{array}{rlrl}
\text { OPTINV }: \max _{T, \lambda} & \sum_{k \in \mathcal{K}} \sum_{a \in \mathcal{A}_{1} \mid x_{a}^{k}=1} n^{k} T_{a}=\sum_{k \in \mathcal{K}} n^{k}\left(\lambda_{d(k)}^{k}-\lambda_{o(k)}^{k}\right) \\
\text { s.t. } & \lambda_{j}^{k}-\lambda_{i}^{k} \leq c_{i j}+T_{i j} & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{1}, \mid x_{i j}^{k}=0 \\
& \lambda_{j}^{k}-\lambda_{i}^{k} \leq d_{i j} & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{2} \mid y_{i j}^{k}=0 \\
& \lambda_{j}^{k}-\lambda_{i}^{k}=c_{i j}+T_{i j} & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{1} \mid x_{i j}^{k}=1 \\
& \lambda_{j}^{k}-\lambda_{i}^{k}=d_{i j} & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{2} \mid y_{i j}^{k}=1
\end{array}
$$

The dual problem of (OPTINV) is:

$$
\begin{array}{rlr}
\text { DOPTINV : } \min _{z} & \sum_{k \in \mathcal{K}} \sum_{(i, j) \in \mathcal{A}} c_{i, j} z_{i j}^{k} & \\
\text { s.t. } & \sum_{(i, j) \in \mathcal{A}} z_{i j}^{k}-\sum_{(j, i) \in \mathcal{A}} z_{j i}^{k}=b_{i}^{k} & \forall i \in \mathcal{N}, \forall k \in \mathcal{K} \\
\sum_{k \in \mathcal{K}} z_{i j}^{k}=0 & \forall(i, j) \in \mathcal{A}_{1} \\
z_{i j}^{k} & \geq 0 & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{1} \mid x_{i j}^{k}=0
\end{array}
$$

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$$
\begin{array}{ll}
z_{i j}^{k} \geq 0 & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{2} \mid y_{i j}^{k}=0 \\
z_{i j}^{k} \quad \text { free } & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{1} \mid x_{i j}^{k}=1 \\
z_{i j}^{k} \quad \text { free } & \forall k \in \mathcal{K}, \quad \forall(i, j) \in \mathcal{A}_{2} \mid y_{i j}^{k}=1
\end{array}
$$

In the case of a single commodity, the above program can be solved through shortest path computations. The efficiency of the method will then rely on efficient shortest path reoptimization procedures [5], which have been adapted to able the detection of negative cost cycles. In the multicommodity case, DOPTINV can be solved efficiently by Dantzig-Wolfe decomposition.

## 3 A local search algorithm

In its local search phase, the algorithm moves to a neighbour of the inducible region that yields a higher leader revenue. It is based on the characterization of feasible followers solutions as a set of commodity paths. The move from a feasible solution to an improved neighbour solution is achieved by forcing the entry of a nonbasic arc in a commodity shortest path tree rooted at the origin node of that commodity, and appropriately removing a basic arc. The evaluation of a move is performed by solving the inverse optimization program described in the previous section.

Two strategies, 'first improvement' and 'best improvement', have been tested. In each of these, a partial evaluation of the current neighborhood is performed. More precisely, the inverse optimization program is solved either on a cluster of commodity (CIO) or by considering each commodity at a time (SIO). The latter strategy is motivated by the efficiency of the solution procedure for the inverse optimization program in the single commodity case.

## 4 Numerical Results

To assess the efficiency of the method, preliminary results on instances that can be solved to optimality using a mixed integer formulation of the (TSP) are provided in Table 1. We consider a set of randomly generated grid networks with 60 nodes $(5 \times 12), 208$ two-way arcs and 10 origin-destination pairs. The proportion of toll arcs varies from $5 \%$ to $10 \%$, and the initial solution is defined as the follower solution corresponding to zero tolls.

We report the results for the two neighborhood evaluation strategies (CIO) and (SIO), together with the exact solution of the MIP formulation using CPLEX. The last line of each subtable contains the average statistics for the corresponding data set, while the first column provides the percentage of toll arcs. The columns 'Time' indicate the computation time in seconds on an Intel 2 GHz Pentium 4 processor. For the Local Search Method, the column labels ' $\% O P T$ ' and ' $\% I / O P T$ ' refer respectively to the ratio of the heuristic objective (respectively the heuristic initial solution) over the optimal solution. When the optimum value is not obtained, it is replaced by the best lower bound achieved. This is denoted by a star $\left(^{*}\right)$ and a nonzero duality gap 'D.G.'. The label ' $\# i t^{\prime}$ refers to the number of iterations.

We observe that the Local Search produces quality solution quite rapidly.
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|  | CPLEX |  |  | CIO |  |  | SIO |  |  |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| \%T | D.G. | Time | \%I/OPT | \%OPT | \#it | Time | \%OPT | \#it | Time |
|  | 0.00 | 3 | 0.81 | 1.00 | 19 | 6 | 0.94 | 14 | 4 |
|  | 0.00 | 4 | 0.75 | 1.00 | 15 | 3 | 1.00 | 20 | 6 |
|  | 0.00 | 10 | 0.82 | 1.00 | 15 | 6 | 1.00 | 13 | 4 |
|  | 0.00 | 4 | 0.61 | 1.00 | 18 | 11 | 1.00 | 13 | 5 |
|  | 0.00 | 4 | 0.99 | 0.99 | 10 | 6 | 0.99 | 10 | 3 |
| 5 | 0.00 | 5 | 0.79 | 0.99 | 15 | 7 | 0.98 | 14 | 5 |
|  | 0.00 | 202 | 0.94 | 1.00 | 15 | 9 | 1.00 | 12 | 3 |
|  | 0.00 | 2987 | 0.60 | 0.97 | 20 | 15 | 0.97 | 23 | 7 |
|  | 0.00 | 194 | 0.94 | 1.00 | 15 | 7 | 1.00 | 18 | 5 |
|  | $* 4.39$ | 7223 | 0.68 | 1.00 | 23 | 21 | 0.91 | 14 | 5 |
|  | 0.00 | 3630 | 0.99 | 1.00 | 16 | 12 | 1.00 | 10 | 3 |
| 10 | 8.17 | 2847 | 0.83 | 0.99 | 18 | 13 | 0.97 | 15 | 5 |

Table 1: Grid Networks with 208 arcs, 60 nodes, 10 commodities

## References

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[^0]:    ${ }^{1}$ The inducible region of a bilevel program consists of feasible solutions whose lower level component is optimal for the lower level problem.

