# A Bilevel Model of Taxation and Its Application to Optimal Pricing of Congested Highways

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## 1 Introduction

We consider the problem of toll setting that faces the operator of a privately owned transportation network in order to maximize its profit. As the users of the network rationally react to the imposed prices, the tolls should be high enough to have a good income but not so much as compelling the users not to use the tolled arcs or the network at all.

In [4] this problem is considered in the case that users choose the shortest route which means that congestion is neglected. This assumption allows them to cast the problem into the class of bilevel optimization problems where both objective functions are bilinear.

In this work we introduce congestion effects through the assumption that travel times depend on the flow and we suppose that the users choose their routes according to the standard Wardrop principle.

We will show that the mathematical problem, which is no longer bilinear-bilinear, can be seen as the maximization of a non-concave function which in general has an infinity of local optima which are very far from the global optimum. Indeed, we study the geometric properties of the upper level objective function defined implicitly as a function of the toll and we show that its non-quasi concavity is an important drawback for standard algorithms.

Nevertheless we prove that the problem has interesting geometric properties that can help in the search for a global optimum. Using an analytical characterization of the lower level solution we determine the cases where the problem is well posed and we propose an algorithm that converges globally to the solution. We illustrate our work with two examples made upon Braess-like networks.

## 2 The model

### 2.1 Traffic assignment

Given a transportation network, the *traffic assignment problem* (TAP) consists in determining a flow that satisfies a given demand between certain pairs of origins and destinations and a certain criterion. Let us consider a traffic network represented by the graph $(\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of directed arcs (links). Each arc $a$ is associated with the positive real number $t_a(f)$ which is the travel time of the link as a function of the network flow $f$. For certain pairs of nodes $(p, q)$ there is a positive flow demand $d_{pq}$ from $p$ to $q$. We call $\mathcal{C} \subset \mathcal{N} \times \mathcal{N}$ the set of origin-destination pairs and we associate each pair with a commodity. The TAP is now to determine a network flow fulfilling the travel demands and a prescribed performance criterion

The performance criteria normally considered are two attributed to Wardrop. The first one, based on the rational behavior of traffic, states that the users seek to minimize their own travel times and it is known as the *user equilibrium.* The second one, known as the *system equilibrium*, calls for the minimization of the total travel time. Even if those criteria are very different (one is behavioral and the other normative) models that consider both criteria have been very well studied in the context of *Marginal Toll Pricing.* Indeed, in this problem, tolls are sought in order that "selfish" routing approach "social" routing (see [2] and [8]).

In this work we focus on the first principle of Wardrop that has the following mathematical formulation (see [5])

$$\begin{array}{rcl} \min T(x) = \sum\limits_{a \in \mathcal{A}} \int_0^{x_a} t_a(s)\, ds\, (x_a) & & \\ \text{s.t.} & & \\ Ax^{pq} & = & \bar{d}_{pq}, \\ x^{pq} & \geq & 0, \\ \sum_{(p,q)\in\mathcal{C}} x^{pq} & = & x. \end{array} \tag{1}$$

where $A$ is the node incidence matrix, $x^{pq}$ is the vector of flows for commodity $pq$, $x$ is the vector of total flows and $\bar{d}$ is the vector of node potentials corresponding to the commodities.

### 2.2 Profit optimization

If a toll $p_a$ (possibly 0) is charged to each car traversing the arc $a$, the total profit will be $p^T x = \sum_{a \in \mathcal{A}} p_a x_a$. If we make the simplifying assumption that tolls are expressed in time units, the travel times changes accordingly being now $t_a + p_a$. So for each given level of tolls there will be a traffic assignment and we can look for the toll that will maximize the total profit. More precisely, we have to find the value of $p$ such that $p^T x(p) = \sum_a p_a x_a(p)$ is maximum where $x(p)$ is the solution of the TAP parameterized by $p$, i.e., the problem (1) where the travel time functions are now $t(\cdot) + p$. As the problem (1) has a unique solution for each $p$, we have that $x(p)$ is differentiable and the convexity of $T$ and the linearity of the constraints also imply that it is convex (see [6]).

# 3 Analytical characterization of the solutions

In order to obtain an analytic formula for the solution of the lower level problem we will suppose that the travel time functions are linear (affine). A more general problem could be then approximated by linear functions and capacity constraints. In this case, using vectorial notation, the problem (1) can be rewritten more compactly as

$$\min_{x \in X} \frac{1}{2} x^T Q x + (b + p)^T x \,,$$

where $Q$ is a diagonal matrix with $Q_{aa} = q_a$ and $b = (b_a)$ a vector such that $t_a(x_a) = q_a x_a + b_a$, $p$ is the vector of prices over the arcs, and $X$ is the polyhedron defined by the constraints in (1), i.e., $X = \{x | Ax = \bar{d},\ x \geq 0\}$.

For $p \in \mathbb{R}^n$ the lower level problem has a unique solution $x(p)$ given by

$$x(p) = \mathrm{Proj}_X^Q(-Q^{-1}(p + b)) = -Q^{-1}[\mathrm{Proj}_P^{Q^{-1}}(p) + b] \,. \tag{2}$$

Thanks to this formula we immediately see that in general for many prices we will have the same distribution of traffic (for any $p \in N_P(\bar{p})$, the normal cone to $P$ in $\bar{p} \in P$), which explains the infinity of local minima (when $x(\bar{p})^\perp \cap N_P(\bar{p})$ is non empty). It can be also proved that the profit function will be the supremum of concave functions. In the figure (1) we present the function $p^T x(p)$ and the subdivision of the $p$-domain such that the function is quadratic and concave (maybe linear) in each region.

Let $P$ be the polyhedron in the price space given by $P = -QX + b$. For $p \in P$, it easily follows that $x(p) = -Q^{-1}(p + b)$ and so the profit function will be quadratic and concave inside $P$. What's more, restricted to $P$, there is a bijective relation between prices and flows and for any $p$ we can obtain a $\bar{p} \in P$ that gives the same profit. The idea behind the algorithm is to exploit this biunivocity between flows and prices, i.e. between $X$ and $P$.

# 4 Numerical applications

Intuitively we can see that when the problem is well posed, each commodity has at least one route without tolled arcs. Otherwise the profit can be made as high as the operator may want. Hence tolls have to be collected only in a subset of arcs or equivalently some prices must be constrained to be zero. Models that consider elastic demand are always well posed, that is clear because to put them in our formalism one artificial arc must be added for each commodity, and these artificial arcs will constitute a non tolled route.