A MULTI-ITEM TRANSPORTATION PROBLEM BY CAPACITATED VEHICLES WITH A JOINT SET-UP COST

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Extended Abstract

Often in supply chain optimization multi-items have to be considered together, due to the dependency of the cost structure or the operational constraints on the total quantities transported and/or replenished. In other cases, the exact composition of the individual items in a single vehicle/batch is important. Other complicating characteristics of multi-item transportation/replenishment problems include capacity limitations of the vehicles/batches, time dependency of demand and cost parameters and the existence of fixed costs per vehicle/batch.

In this paper we consider a multi-item lot-sizing problem in which transportation (production) takes place in mixed vehicles (batches) of constant capacity. Apart from the usual unit storage and production costs, there is a fixed cost per vehicle (batch) representing the use of limited capacity resource. In the transportation context, a warehouse ships a number of items by using trucks of a given capacity so as to (possibly) stock items and satisfy demand forecasts at the retailer. The objective is to find replenishment decisions for all items in order to satisfy the demand for the items over a finite planning horizon, and minimize the sum of fixed costs associated with the vehicles used, variable production costs and storage costs.

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This problem is a member of the dynamic lot-sizing problems with capacities which are of the most frequently used deterministic inventory planning problems. In addition to being directly representative of various transportation and production settings, they often arise as sub-problems of more complex multi-echelon dynamic lot-sizing problems when those are decomposed as part of the solution method.

However, dynamic lot-sizing problems with capacity restrictions are known to be hard problems. When only one vehicle (batch) is allowed in each period and the capacity limitation is time-dependent, even the single item case is known to be NP-Hard, see Florian, Lenstra and Rinnooy Kan (1980). When the capacity limitations are time independent, the single item problem is solved in polynomial time, see Florian and Klein (1971) and Van Hoesel and Wagelmans (1996). Pochet and Wolsey (1993) considered the single item problem with multiple batches, time-varying set-up, inventory holding and variable production costs. They designed an $O(T^3)$ algorithm, where T is the number of periods, which is based on finding a shortest path in an appropriately defined network. Lee (1989) addressed the single item multiple batch problem in which there exists a setup cost for ordering in a particular period, in addition to a different setup cost incurred for each batch, and presented an $O(T^4)$ procedure to solve it. Results concerning polynomial algorithms for multi-item problems are limited. Exceptions are the multi-item discrete lot-sizing problems with and without backlogging in which a limited number of items are produced per period, see Miller and Wolsey (2003b).

In this paper we consider the problem with multiple items and multiple vehicles/batches, that was studied recently by Anily and Tzur (2004,2005), under the following restrictions:
i) the unit production costs of each item are stationary over time. It follows that the total production costs are constant, and can thus be ignored.
ii) the storage costs for each item are stationary over time and non-negative.

Under these assumptions, Anily and Tzur (2005) develop a dynamic programming algorithm whose running time is polynomial for a fixed number of items. More precisely, the running time of their algorithm is O(mT^{m+5}), where m is the number of items, and T is the number of periods. Anily and Tzur (2004) propose an alternative optimal search algorithm and heuristics for solving the problem. See the references therein for additional background on the problem and a literature review on single and multiple item capacitated dynamic lot-sizing problems, with and without batching considerations.

Here we consider essentially the same problem, except that ii) is relaxed

to ii') and iii'):
ii') the storage costs of each item in each period are required to be non-negative. In view of the fact that the production costs are time independent, the non-negativity of the storage costs results in the *Wagner-Whitin* cost property or *non-speculative costs* for each item.
iii') there exists a permutation Π of the set of items such that the $k - th$ item in the permutation is the $k - th$ most expensive item to store in all periods. In other words, there exists an indexing of the items according to the non-increasing ordering of their storage costs that is preserved in all time periods. We call the resulting problem $WW^* - CC - FAM$.

Under these generalized conditions we derive a tight and compact extended formulation for the problem. Here a formulation is *extended* when it involves additional variables, but its projection back into the original space along with the integrality constraints gives precisely the MIP feasible region. It is *compact* if its size is polynomial in the size of the input, and it is *tight* when its projection is precisely the convex hull of the MIP feasible region.

It follows, because it is compact, that solving a linear program over the extended formulation can be carried out in polynomial time. What is more, because it is tight, an optimal extreme point solution can be found that is feasible and optimal for the MIP. Thus the problem is in $\mathcal{P}$. The effectiveness of this approach has been demonstrated on a wide variety of lot-sizing problems starting with Eppen and Martin (1987). The forthcoming book of Pochet and Wolsey (2006) classifies and presents the state-of-the-art on the formulation and solution of a wide variety of production planning problems by mixed integer programming, and demonstrates the effectiveness of tight extended formulations on several industrial cases.

The contributions of this paper are as follows: we present a linear program with $O(mT^2)$ constraints and variables that solves problem $WW^* - CC - FAM$, thus showing that it is polynomially solvable for any number of items and periods. Alternatively, the problem can be solved as a linear program with $O(mT)$ constraints and variables combined with an $O(mT^2 \log T)$ separation algorithm. This resolves a long standing open question concerning the complexity of capacitated multi-item lot-sizing problems where the items share a common set-up, see Bitran and Yanasse (1982). If backlogging is allowed, a similar result is shown to hold for the problem in which production is uncapacitated, the storage costs are constant over time, and the backlogging costs are a fixed multiple of the storage costs. Computationally we demonstrate numerically the effectiveness of the extended formulations relative to the original formulations.

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