Existence, uniqueness and stability of equilibrium in dynamic traffic networks

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Abstract

In the steady state model, costs at user equilibrium are unique and the system converges to user equilibrium. In the dynamic model, there are various traffic performance models. Our focus here is on the bottleneck queueing model; although the results can be applied to any dynamic model satisfying the stated properties. The bottleneck queueing model has deterministic queueing at link exits when flow exceeds capacity. Within-day time is considered as continuous, and flow rates into routes connecting origin-destination (OD) pairs are represented by functions of time. The day-to-day route-swapping is modelled by a continuous dynamical system that is derived naturally from the dynamic user equilibrium condition. Demand for travel between each OD pair in the network is considered to be rigid, in the sense that it is time-varying within-day but unchanging from day to day.

Existence of equilibrium is guaranteed provided that the route cost vector is a continuous function of the route flow vector, which holds in the bottleneck model. Costs at dynamic user equilibrium are unique in the single OD pair case provided that certain (natural) non-decreasing properties are satisfied; the bottleneck model satisfies these properties. It seems likely that this could be extended to the single origin case and the single destination case. The dynamical system simulating a swap to less costly routes converges to equilibrium if the route cost is either monotone or decay-monotone; essentially this is true in the single bottleneck per route case, but not in the multiple bottlenecks per route case. In the multiple origins and multiple destinations case, dynamic user equilibrium may not be unique; there may be separate equilibria each having their own respective domains of attraction. In this case any optimisation strategy would have to guide the system towards a more favourable equilibrium.

1. Introduction

There are a number of ways to model the dynamic flow of traffic in networks. These include the outflow model that was proposed by Merchant and Nemhauser [1978], which considers the outflow from a link to be entirely dependent upon the amount of traffic on the link. Another approach proposed by Lighthill and Whitham [1955] and Richards [1956] is to model flow as one-dimensional compressible fluid, which is approximated in discrete space and time by the cell transmission model (Daganzo [1994]). The model that we will adopt here is a deterministic queueing model.

This type of model was considered by Smith and Ghali [1990a,1990b] and Bernstein et al. [1993], and later expanded upon by Smith and Wisten [1995] and Mounce [2006]. In such a model, queueing occurs in a first in first out (FIFO) manner vertically at the exit of links when flow exceeds capacity. It is possible to model queues forming backwards and potentially spilling back onto other links (see e.g. Adamo et. al. [1999] and Kuwahara and Akamatsu [2001]), which is a more realistic model of road traffic, but also more complex.

Traffic assignment models must take account of the behaviour of travellers in the network. The system optimal problem seeks to minimise the total cost of all travellers throughout the network. A model for approximately solving the dynamic system optimal problem is given by Ghali and Smith [1995]. In this paper, we assume that travellers make routing decisions aimed at reducing their own travel costs. In the steady state case, this is quite straightforward to formulate (see e.g. Wardrop [1952]) but in the dynamic model there are different interpretations. In the instantaneous cost path problem, travellers attempt to minimise the instantaneous cost (the sum of link costs along the route evaluated at the current time) to their destination. They may later revise their route if a new path becomes less costly (instantaneously). The model we adopt for route choice is based on dynamical user equilibrium, where more costly routes are unused for all within-day time. Perceived cost is the cost actually experienced by a traveller traversing the route, and a traveller cannot change their route once they have left their origin. They may, however, change their route for tomorrow if there is one available that was today less costly.

2. The steady state model

In the steady state model, the cost to traverse each link $c_i(x)$ is a real-valued function of the link flow x_i (also real-valued). Link flows are found by summing route flows, denoted X_r, over all of the routes traversing that link, i.e.

$$x_i = \sum_{r:i\in r} X_r$$

where $i \in r$ means that route r traverses link i. Route costs, denoted $C_r(X)$, are found by summing link costs, i.e.

$$C_r(X) = \sum_{i:i\in r} c_i(x)$$

If all link cost functions are non-decreasing functions of link flow, then the route cost vector is a non-decreasing function of the route flow vector (this will be defined precisely in Section 4). Minty's Lemma then tells us that the set of equilibria is convex. Mounce and Smith [2007] show that costs at user equilibrium are unique in this case. If each link cost is an increasing function of link flow, then the equilibrium link flows are unique. If each link cost is a non-decreasing function of link flow, then the costs at equilibrium are unique. Given any starting route flow vector, the dynamical system modeling a swap to less costly routes converges to the set of equilibria (Smith [1984]).

3. The dynamic model

The traffic network is considered to be a directed graph where traffic flows along acyclic directed paths, called routes, connecting origins to destinations. Within-day time is considered to be continuous and varying within the interval [0,1]. Traffic flow is considered to be a continuous variable and the flow rate into route r, denoted X_r, is a real-valued function of within-day time t in $[0,1]$ - this function may or may not be continuous, but it will be a non-negative, measurable and essentially bounded function of time t. The null sets are quotiented out, leaving equivalence classes of functions equal almost everywhere (unequal only on a null set). This means that each route flow function belongs to the space $L^\infty[0,1]$. If there are N routes in the network, then these route flow functions combine to give the route flow vector $\mathbf{X}$, i.e. $\mathbf{X} = (X_1, X_2, \ldots, X_N)$, which is in the space $\oplus_{i=1}^N L^\infty[0,1]$.

The demand for travel between OD pair k, denoted by ρ_k, is supposed rigid, i.e. it is a fixed element of $L^\infty[0,1]$ for each k. The route flow vector $\mathbf{X}$ belongs to the feasible set D, consisting of all those vectors in $\oplus_{i=1}^N L^\infty[0,1]$ that have non-negative components and also meet the given rigid demand for travel between each OD pair. Hence, if R_k to be the set of all routes joining OD pair k,

$$D = \{X \in \oplus_{i=1}^N L^\infty[0,1] : X_r \geq 0 \;\forall\, r \;\&\; \sum_{r \in R_k} X_r = \rho_k\}.$$

Given any link inflow function x_i, the cost to traverse link i if entered at time t is the sum of a constant (congestion-free) travel time c_i, a constant (monetary) price p_i (that will be converted into a cost in time units) and a bottleneck delay $d_i^x(t)$ (depending on the whole link flow function). Let s_i represent the capacity flow rate at the exit of link i. When traffic flow exceeds this capacity, a queue starts to form at the exit of the link. The bottleneck is said to be congested when the queue (and hence the delay) is positive. The bottleneck delay at link i is connected to the bottleneck capacity s_i and the bottleneck inflow x_i by the following integral equation:

$$\int_{t_0}^{t} x_i(u)du = \int_{t_0}^{t+d_i^x(t)} s_i(u)du. \tag{1}$$

for all t in some congested period $[t_0, t_1]$ (i.e. the bottleneck becomes congested immediately after time t_0 and remains congested until time t_1).

Bottleneck queueing is first-in first-out (Mounce [2007]), i.e. if traveller A enters the link before traveller B, then traveller A must also exit the link before traveller B. The cost to traverse a route is the sum of the link costs at the time each link is reached, i.e.

$$C_r^X(t) = \sum_{i:i\in r} c_i^x(A_{ir}^X(t))$$

where $A_{ir}^X(t)$ is the time that link i is reached if route r is entered at time t and the route flow vector is $\mathbf{X}$. The route cost vector is defined by $\mathbf{C}(\mathbf{X}) = (C_1^{\mathbf{X}}, C_2^{\mathbf{X}}, \ldots, C_N^{\mathbf{X}})$, which is in $\oplus_{i=1}^N C[0,1]$ (Mounce [2006]). In order to consider continuity of the route cost vector with respect to the route flow vector, it is necessary to introduce appropriate measures of distance on the set of feasible route flow vectors and on the set of feasible route cost vectors. The norm on the set of feasible route flow vectors is determined by the supremum norm of the cumulative route flows, so that the distance between two feasible route flow vectors $\mathbf{X}$ and $\mathbf{Y}$ is given by

$$d(\mathbf{X}, \mathbf{Y}) = \sup_r \sup_{t \in [0,1]} \left| \int_0^t (X_r(u) - Y_r(u)) du \right| .$$

The norm on the set of route costs is the supremum norm, i.e.

$$\|C(X)\| = \sup_r \sup_{t \in [0,1]} \left| C_r^X(t) \right| .$$

The model adopted for route choice is based on dynamical user equilibrium, where more costly routes are unused for all within-day time (an extension of Wardrop's [1952] definition of user equilibrium to the dynamic model). A traveller cannot change their route once they have left their origin, but they may change their route for tomorrow if there is one available that was today less costly. The dynamical system evolves continuously (in this case) over day-to-day time with each element being a within-day flow pattern. Given a route-flow vector $\mathbf{X}$, define the day-to-day route swap vector $\phi(\mathbf{X})$, for each $t \in [0, 1]$, by

$$\phi(\mathbf{X})(t) = \sum_{r,s:r\sim s} X_r(t)[C_r^X(t) - C_s^X(t)]_+ \delta_{\mathbf{rs}} \tag{2}$$

where $r \sim s$ if r and s connect the same OD pair, $x_+ = \max\{0, x\}$ and $\delta_{\mathbf{rs}}$ is the swap from route r to route s vector (i.e. it is an N-vector with -1 in the rth place and 1 in the sth place and zeros elsewhere). Then let τ represent day-to-day time, and consider the dynamical system

$$\begin{aligned} \frac{d\mathbf{X}(\tau)}{d\tau} &= \phi(\mathbf{X})(\tau), \\ \mathbf{X}(0) &= \mathbf{X_0}, \end{aligned} \tag{3}$$

for $\tau \geq 0$ and where $\mathbf{X_0} \in D$. A state $\mathbf{X}^*$ is now termed an equilibrium of (3) if and only if $\phi(\mathbf{X}^*) = 0$, i.e. the dynamical system remains in the state $\mathbf{X}^*$ for all future time if and when it is reached. Naturally, dynamical user equilibrium is such that for all within-day time and all OD pairs, more costly routes are not used, and therefore there is no incentive to change route, i.e. for any routes r and s connecting the same OD pair, $C_r^X(t) > C_s^X(t) \implies X_r(t) = 0$ for all within-day time t. Clearly, dynamical user-equilibrium coincides with an equilibrium of the dynamical system (3). If, for each choice of initial state $\mathbf{X_0} \in D$, there exists an $\mathbf{X}^*$ such that $\mathbf{X}(\tau) \to \mathbf{X}^*$ as $\tau \to \infty$, the dynamical system is said to be *globally convergent*.

4. Monotonicity

In the steady state model, the link cost $c_i(x_i)$ is a non-decreasing function of the link flow x_i if and only if $c_i(x_i) \geq c_i(y_i)$ whenever $x_i > y_i$, or equivalently

$$(c_i(x_i) - c_i(y_i))(x_i - y_i) \geq 0 \tag{4}$$

for all link flows x_i and y_i. Link cost is an increasing function of link flow if strict positivity holds in (4) whenever $x_i \neq y_i$. Generalising this to vectors, the link cost vector $\mathbf{c}(\mathbf{x})$ is a non-decreasing function of the link flow vector $\mathbf{x}$ if and only if (where the dot represents the vector dot product)

$$(\mathbf{c}(\mathbf{x}) - \mathbf{c}(\mathbf{y})) \cdot (\mathbf{x} - \mathbf{y}) \geq 0 \tag{5}$$

for all link flow vectors $\mathbf{x}$ and $\mathbf{y}$.

In the dynamic case, the route cost vector is an operator, i.e. the route cost vector function depends upon the whole route flow vector function. A natural generalisation of (4) in the dynamic case is to say that:

Definition 1. The bottleneck delay d_i^x is a monotone function of x_i if and only if

$$\int_0^1 (d_i^x(t) - d_i^y(t))(x_i(t) - y_i(t))dt \geq 0$$

for all link flows x_i and y_i.

Only the bottleneck delay varies with the link flow, so that link cost is a monotone function of link flow if and only the link delay is a monotone function of link flow. A natural generalisation of (5) in the dynamic case is to say that:

Definition 2. The route cost vector $\mathbf{C}(\mathbf{X})$ is a monotone function of the route flow vector $\mathbf{X}$ if and only if

$$\sum_r \int_0^1 (C_r^X(t) - C_r^Y(t))(X_r(t) - Y_r(t))dt \geq 0$$

for all route flow vectors $\mathbf{X}$ and $\mathbf{Y}$.

In the single bottleneck per route case each route passes through at most one active bottleneck (i.e. a link that is congested for some within-day time). Smith and Ghali [1990] showed that in the single bottleneck per route case, route cost is a monotone function of route flow if link cost is a monotone function of link flow for every link. This is trivially true for non-bottleneck links. Mounce [2006] showed that the delay function is a monotone function of the flow into the bottleneck if and only if the bottleneck capacity is a non-decreasing function of within-day time. Therefore, in the single bottleneck per route case with non-decreasing capacities at each bottleneck, the route cost vector is a monotone function of the route flow vector. If route cost is monotone, Minty's Lemma can be used to show that the set of equilibria is convex (Mounce [2007]). However, in the multiple bottleneck per route case, route cost is not generally monotone (Mounce and Smith [2007]). It is also possible to define the route cost vector as a non-decreasing function of the route flow vector in terms of a partial order (Mounce and Smith [2007]).

5. Decay-monotonicity

Link delay fails to be a monotone function of link flow if the bottleneck capacity is not a nondecreasing function of within-day time. However, even in this case, when cumulative link flow increases on an interval, link delay does increase; at least on some future time interval. This notion is captured by the property of decay-monotonicity defined below:

DEFINITION 3. Bottleneck delay d_i^x is a *decay-monotone* function of link flow x_i if and only if there exists a constant $k \in \mathbb{R}$ such that

$$\int_0^1 (d_i^x(t) - d_i^y(t))(x_i(t) - y_i(t))e^{-kt}dt \geq 0.$$

for all link flow functions x_i and y_i.

The bottleneck delay is a decay-monotone function of the bottleneck link flow providedthat the bottleneck capacity is continuously differentiable (Mounce [2008]). In practice this means that the bottleneck capacity is in general decay-monotone since any piecewise continuous function can be approximated arbitrarily closely by a continuously differentiable function.

DEFINITION 4. The route cost vector is a *decay-monotone* function of the route flow vector if and only if there exists a constant $k \in \mathbb{R}$ such that

$$\sum_r \int_0^1 (C_r^X(t) - C_r^Y(t))(X_r(t) - Y_r(t))e^{-kt}dt \geq 0 \tag{6}$$

for all route flow vectors $\mathbf{X}$ and $\mathbf{Y}$.

If each route passes through at most one bottleneck, with the travel time to the bottleneck constant for all routes passing through it, then the route cost vector is a decay-monotone function of the route flow vector if and only if link cost is a decay-monotone function of link flow at every link. This is a restricted version of the single bottleneck per route case; and unlikely to be satisfied in practice. It remains to be seen whether the decay-monotonicity property will prove of more use in tackling convergence in dynamic traffic networks in a more general context.

6. Existence of equilibrium

Smith and Wisten (1995) proved that existence of dynamic user equilibrium is guaranteed if the route cost vector is a continuous function of the route flow vector. This was achieved by applying Schauder's fixed point theorem to the map T defined by

$$T(\mathbf{X}) = \mathbf{X} + \alpha\phi(\mathbf{X})$$

(where α is a small constant) to give the existence of a fixed point of T; and therefore an equilibrium of the dynamical system (3).

Mounce [2006] proved that the link delay function (and hence the link cost function) is a continuous function of link flow. Mounce [2007] proved that the route cost vector is a continuous function of the link cost vector and that the link flow vector is a continuous function of the route flow vector; this second result was achieved by defining a function of the link inflows and link outflows (which vary independently of each other) that is Lipschitz continuous to which an Implicit Function Theorem was applied to prove that link outflows are a continuous function of link inflows. Therefore the route cost vector is a continuous function of the route flow vector (in the bottleneck model) and existence of equilibrium is guaranteed by Smith and Wisten [1995].

7. Uniqueness of equilibrium

When route cost is monotone, the set of equilibria is convex (Mounce [2007]). So in this case, we know that there cannot be two disconnected equilibria. However, in the dynamic model, route cost is not in general monotone. Mounce and Smith [2007] use a partial order to define the notions of increasing and non-decreasing as follows:

DEFINITION 5. Link cost c_i^x is a non-decreasing function of link flow x_i if and only if whenever $\int_t^{t'} x_i(u)du \geq \int_t^{t'} y_i(u)du$ for all $t, t' \in [0,1]$ with $t \leq t'$, we have $c_i^x(t) \geq c_i^y(t)$ for all $t \in [0,1]$.

Costs at dynamic user equilibrium are then shown, using a proof by contradiction, to be unique in the single OD pair case provided that link cost is a non-decreasing function of link flow, link outflow is a non-decreasing function of link inflow, and certain other natural non-decreasing properties are satisfied; the bottleneck model is shown to satisfy these properties. In the bottleneck model, if costs are unique then link flows are unique when there is congestion. Uniqueness of equilibria is also likely in the single origin case and in the single destination case; although this has not yet been shown. In networks with both multiple origins and multiple destinations, dynamic user equilibrium may not in general be unique; there may be disconnected equilibria.

8. Convergence to equilibrium

The dynamical system (3) is globally convergent to equilibrium if for each choice of initial state $\mathbf{X_0} \in D$, there exists an equilibrium vector $\mathbf{X}^*$ such that $\mathbf{X}(\tau) \to \mathbf{X}^*$ as $\tau \to \infty$, i.e. the system gets closer and closer to equilibrium as day-to-day time goes by. A standard method of proving global convergence to equilibrium is to define a Lyapunov function, as in the following theorem:

THEOREM 1. The dynamical system (3) is globally convergent to equilibrium if there is a continuous scalar-valued function $V(\cdot)$ defined throughout D such that:

(1) $V(\mathbf{X}) \geq 0$ for all $\mathbf{X}$ in D,
(2) $V(\mathbf{X}) = 0$ if and only if $\mathbf{X}$ is an equilibrium, and
(3) $\frac{dV(\mathbf{X})}{d\tau} < 0$ for all non-equilibrium $\mathbf{X}$.

A proof of this result in the steady state model is given in Smith [1984] and a proof in the dynamic model is given in Mounce [2008]. When route cost is monotone, the following theorem can be applied:

THEOREM 2. If the route cost vector is a monotone function of the route flow vector, then

$$V(\mathbf{X}) = \sum_{r,s:r\sim s} \int_0^1 X_r(t)(C_r^X(t) - C_s^X(t))_+^2 dt$$

is a Lyapunov function for the dynamical system (3) and therefore the dynamical system (3) is globally convergent to equilibrium in this case.

PROOF. See Mounce [2006]. □

Route cost is not in general monotone in the multiple bottlenecks per route case (Mounce and Smith [2007]). Mounce [2001] gives a simple example network where non-monotonicity is utilised in showing that the Lyapunov function in Theorem 2 is not a Lyapunov function, in general, for the multiple bottlenecks per route case.

When route cost is decay-monotone, the following theorem applies:

THEOREM 3. If the route cost vector is a decay-monotone function of the route flow vector, then

$$V(\mathbf{X}) = \sum_{r,s:r\sim s} \int_0^1 X_r(t)(C_r^X(t) - C_s^X(t))_+^2 e^{-kt} dt$$

is a Lyapunov function for the dynamical system (3).

PROOF. See Mounce [2008]. □

9. Conclusion

In the bottleneck model, existence of equilibrium is guaranteed since the route cost vector is a continuous function of the route flow vector. When the route cost vector is a monotone function of the route flow vector, it follows that the set of equilibria is convex and the dynamical system simulating a swap to less costly routes converges to equilibrium for all starting route flow vectors. Convergence also follows when the route cost vector is a decay-monotone function of the route flow vector. In general, route cost is monotone in the single bottleneck per route case but not in the multiple bottlenecks per route case. However, costs at equilibrium are unique in the single OD pair case; this result may well be extended to the single origin case and the single destination case. However, in general, there may be separated equilibria.

The results in the paper require certain properties of the dynamic traffic assignment model. The bottleneck model satisfies these properties; but these properties may well be satisfied by other traffic assignment models.

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