Applications of Vehicular Traffic Theory to Blood Rheology

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1 Introduction

For decades, blood rheology has been studied both theoretically and experimentally. In most cases, blood has been modelled as a homogeneous fluid described by classical hydrodynamic equations. An overview of recent numerical methods for modelling vascular flow on a large scale can be found in Quarteroni (2002). We develop a framework for modelling vascular blood flow using the vehicular traffic approach, and show the theoretical and computational gains it yields. We compare this approach to the more complex and computationally intensive Boltzmann-type equation models used by other blood rheology researchers (Dzwinel et al., 2003, Sun and Munn, 2005, Mehrotra et al., 1985), and prove the potential for better leverage of ultrasonography data. As a useful example, we investigate the red blood cells' clustering phenomena within the Payne-Whitham model.

We describe the flow of blood in a linear, although sufficiently wide, vessel. More precisely, we consider the movement of numerous blood particles (e.g. red blood cells and white blood cells), varying in size and shape, along such vessel. In a typical laminar flow, these particles move along parallel lines following each other, although some mixing between them is also allowed. There exist at least three approaches to the multi-lane vehicular traffic problem. The first approach concerns microscopic and follow-the-leader approaches, modelling the actual response of a single car to its predecessors. On the other hand, macroscopic (fluid dynamics) approaches constitute a direct extension of the Payne-Whitham model for linear traffic flow or, alternatively, otherwise employ classical hydrodynamics to traffic modelling. Kinetic (Boltzmann-like) models present an intermediate step between the above two approaches, and appears to be the most promising for our problem, although relatively complicated. The kinetic model description uses the distribution function in the system phase space rather than a description of individual cars.

We propose a kinetic framework for the blood rheology problem, which is inspired by vehicular traffic models (Illner et al., 2002). In this framework, particles move along parallel lines with their speed dependent on the density of the surrounding particles. Particles may also switch between these lines, if the density conditions in the other line are more favorable (i.e. density is lower). Their movement is then governed by Vlasov partial differential equation (PDE). Vlasov-type kinetic models are capable of representing the various phases and phase transitions observed in vehicular traffic flow by Kerner and Konhauser (1994), including free flow, synchronized flow, synchronized equilibria, and congested flow. Our purpose is to show that the same approach is extremely suitable for blood rheology, and is capable of reproducing the phenomena observed in blood flow (Dzwinel et al., 2003). It is worth mentioning that our traffic-inspired models, while significantly simplifying the problem treatment, seem to outperform or be equivalent to most microscopic models proposed by Dzwinel et al. (2003), Sun and Munn (2005), and others.

Our basic framework is as follows. Consider a linear blood vessel (e.g. artery) with obstacles (cholesterol buildup) along its length (fig. 2). Using ultrasonography techniques, we can measure the blood velocity $v(x)$ as a function of position along the vessel. We consider a stationary situation, which implies that the flow is constant along the vessel. Now, we make the assumption that the blood flow is similar to the vehicular traffic continuous flow model. More precisely, we assume that there exists an unambiguous "fundamental flow diagram" — an explicit relationship between density and velocity of the flow — which itself depends on certain physical parameters (for instance, the classical Whitham-Lighthill fundamental flow diagram can be parameterized by the maximum throughput density ρ_{max}).

If we apply our fundamental flow diagrams family, developed within our kinetic traffic flow model, to the ultrasonographic measurements, we can deduce the diagram parameters as functions of position along the vessel x. The parameters, in their turn, have direct dependence on the vessel radius (in the classical Whitham-Lighthill diagram, ρ_{max} may be a function of radius $R(x)$). This procedure, therefore, allows us to deduce the $R(x)$ vessel profile from the velocity measurements, and detect the obstacles in the vessel in an automated fashion, which is much more accurate than conventional observation techniques.

A more involved and even more accurate procedure is the following. We reject the assumption of constant flow, but instead consider a time-dependent (due to cyclic heart pumping) flow in the one-dimensional vessel using the Payne-Whitham model, equipped with the same fundamental flow diagram of the kinetic traffic model, and with a periodic source as a boundary condition. In this case, we need to match the time-dependent velocity field to the results of the Payne-Whitham simulation, and solve for the matching parameters (eventually, radius $R(x)$).

The problem we are concerned with may thus be divided into the following topics under investigation (fig. 1).

A- Application of the Payne-Whitham single-lane traffic model to the arterial blood flow, given the fundamental diagram. Given the fundamental flow diagrams family, developed within the kinetic traffic flow model, to the ultrasonographic measurements, we address the problem of retrieving the vessel profile from the velocity measurements, and detecting obstacles in the vessel.

B- Derivation of the fundamental flow diagram using the kinetic multi-lane traffic

model. In this problem, we deduce the fundamental flow diagram for a predefined set of regime parameters by obtaining the distribution function f_i as a solution to Vlasov PDE.

C- Inverse problem of determination of physical blood characteristics from the fundamental diagram derived empirically. We study the dependence of flow (and the respective fundamental flow diagram) on the vessel internal geometry and parameters such as blood viscosity.

D- Use of experimental (ultrasonographic) velocity/density measurements to derive fundamental flow diagram.

E- Modelling of red blood cell clustering in blood vessels within the Payne-Whitham framework. Cluster solutions have been observed in vehicular traffic; we investigate the solutions to the Payne-Whitham model with a flow profile appropriate in blood rheology, to reproduce the red blood cells' clustering phenomena by appropriate calibration of the flow profile.

F- Inverse problem of determination of obstacles' profile in a stenosed artery.

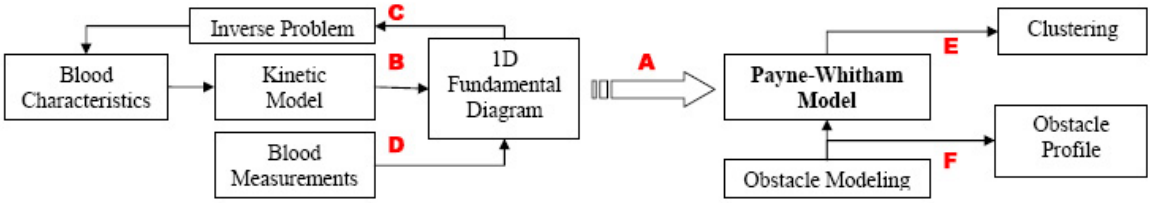

Figure 1: Framework for applying vehicular traffic theory to blood rheology.

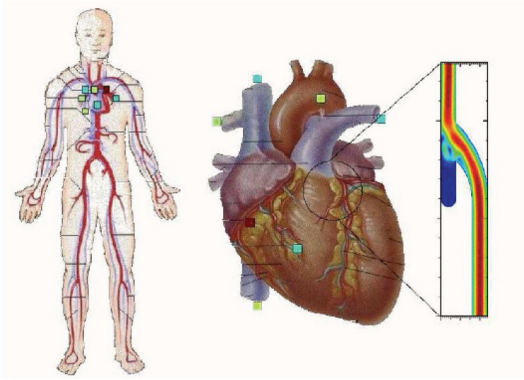

Figure 2: A typical linear vessel link considered (from Quarteroni, 2002)

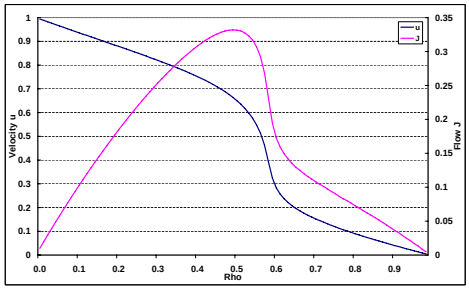

Figure 3: Fundamental flow diagram derived using the (forward) procedure described in section 4, $C_a = 5.0, C_b = 7.0, C_\sigma = 1.5$.

2 Payne-Whitham single-lane model

In this section, we address problem A outlined above. According to the Payne-Whitham model, non-equilibrium traffic on a long one-dimensional path may be described by

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ j\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}j\\ \frac{j^2}{\rho}+\kappa\rho\end{pmatrix}=\begin{pmatrix}0\\ \frac{\rho v_*(\rho)-j}{\tau}\end{pmatrix}, \tag{1}$$

the system of hyperbolic conservation laws with relaxation. Here $v_*(\rho)$ is given by the "fundamental diagram", $\rho(t,x)$ is the traffic density at time t and location x, $j(x,t)=\rho v$ is the traffic flow, and κ and τ are fluid parameters.

The core idea in our analysis of this problem is to use the first-order Godunov finite difference scheme to model the flow numerically. In this scheme, the finite-difference solution on the rectangular grid is given by

$$\rho_i^{k+1}=\rho_i^k-\frac{\delta}{\hbar}(\hat{j}_{i+1/2}^k-\hat{j}_{i-1/2}^k), \tag{2}$$

$$j_i^{k+1}=\frac{1}{1+\delta/\tau}(j_i^k-\frac{\delta}{\hbar}\left(\frac{(\hat{j}_{i+1/2}^k)^2}{\hat{\rho}_{i+1/2}^k}-\frac{(\hat{j}_{i-1/2}^k)^2}{\hat{\rho}_{i+1/2}^k}+\kappa\hat{\rho}_{i+1/2}^k-\kappa\hat{\rho}_{i-1/2}^k\right)+\frac{\delta}{\tau}\rho_i^{k+1}v_*(\rho_i^{k+1})). \tag{3}$$

Here, i and k are the spatial and temporal index respectively; δ and $\hbar$ are the grid steps in time and space respectively, and $\hat{U}_{i+1/2}^k=(\hat{\rho}_{i+1/2}^k,\hat{j}_{i+1/2}^k)$ is the solution to the original PW system with initial conditions ($U_i^k=(\rho_i^k,j_i^k)$)

$$u_{i+1/2}(x,k\delta)=\begin{cases}U_i^k, & x<\hbar(i+\frac{1}{2}),\\ U_i^{k+1}, & x\geq\hbar(i+\frac{1}{2}),\end{cases} \tag{4}$$

which can easily be expressed analytically.

Jin and Zhang (2002) showed the convergence of the Godunov finite difference scheme for initial conditions inside the stability domain of the Payne-Whitham model. They also showed that the numerical method would not yield convergent solutions for unstable initial conditions, since the PW model, when it is unstable, is very sensitive to small changes in initial conditions.

Having developed the approach to modelling single-lane flow, using the appropriate fundamental flow diagram profile, we apply this approach to different initial conditions and investigate the properties of the resulting blood flow.

3 Multi-lane model for the fundamental diagram derivation

In this section, we define the generalization of a kinetic model proposed for vehicular traffic flow by Illner et al. (2003). We consider a linear blood vessel with a round cross-section, which we divide into concentric tube regions with equal cross-section areas; these regions play the

role of traffic lanes. By $f_i(t, x, v)$ we denote the particle distribution function at time t, at location x along the vessel, and velocity v in the lane i, $i = 1, 2, \ldots$. The governing Vlasov equation has the form

$$\partial_t f_i + v\partial_x f_i + \partial_v \left(B[f_i]f_i - D[f_i]\partial_v f_i \right) = \sum_{l \neq i} p_{li}[f_l]f_l - p_{il}[f_i]f_i, \tag{5}$$

where $B[f_i]$ is the "acceleration/deceleration" term, corresponding to $\ddot{x} = B[f_i]$ for a given particle, $D[f_i]$ is the diffusion coefficient, $p[f_i]$ is the "switching" rate,

$$p_{il}[f_i] = j_i(t, x)P_{il}(u_i, v), \tag{6}$$

where P_{il} is the switching probability from lane i to l, $u_i = j_i/\rho_i$ is the average lane speed. The essence of the model lies in the assumptions regarding the functional forms of these terms.

For instance, we assume that the switching probability from lane i to lane l depends on the velocity in lane i in the following way:

$$P_{il}(u, v) = \begin{cases} \left(\frac{v-u}{1-u}\right)^{\delta}, & v > u, l = i \pm 1, \\ 0, & v \leq u \end{cases} \tag{7}$$

In our analysis, by obtaining the distribution function f_i as a solution to (5), we deduce all possible macroscopic quantities of interest, and deduce the fundamental flow diagram for the predefined set of regime parameters. Our goal here is to determine common characteristics and profiles in the fundamental flow diagrams in patients within similar age group, and with similar pathological cardiovascular conditions.

4 Applications to blood flow modelling

We apply the traffic model described above to real ultrasonography data we collect from clinical data on both humans and animals, and compare their modelling capabilities.

4.1 Dependence on flow parameters

An example of the application of the procedure described above including the numerical solution is shown in figure 3. In general, the numerical simulations lead to the following observations regarding the inverse problem of solving for parameters.

The fundamental diagram solution $u(\rho)$ always exists, but may not be unique. Intervals of the parameters C_a, C_b, C_σ that lead to unique fundamental diagrams may easily be established by direct computation. Parameters C_a, C_b, C_σ may be estimated (backwards) from the fundamental flow diagram (in fact, just from its special points, e.g., from its maximum $j_{\max}$). We derive necessary and sufficient conditions for the maximum flow $j_{\max}(C_a, C_b, C_\sigma)$ to be invertible.

4.2 Application to red blood cells clustering

We provide an application of the vehicular traffic theory to a topic that has recently received attention in blood rheology circles — clustering of red blood cells. When the Payne-Whitham model is unstable, Kerner and Konhauser (1994) observed cluster solutions in vehicular traffic. Our goal is to investigate, both numerically and analytically, solutions to the Payne-Whitham model with a specific flow profile, appropriate in blood rheology, to show the solution patterns of the model when initial conditions are unstable, to replicate the clustering phenomena by appropriate calibration of the flow profile.

We model the red blood cells' clustering phenomena within the Payne-Whitham model in the unstable regime. In the case of an obstacle on a one-dimensional path (e.g., a deformed wall), the conditions on a certain interval of the line change, which effectively changes the fundamental diagram for the interval. The solution for this problem is piecewise-continuous, starting with an initial piecewise-continuous density profile as an initial condition. In our study, we derive explicit analytical solutions for a number of initial flow profiles.

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