ON NUMERICALLY SOLVING
CONTINUOUS SPACE DYNAMIC ROUTE CHOICE PROBLEMS

Hoogendoorn, S.P., W. Daamen, and M. Campanella
Transportation and Planning Department, Delft University of Technology, The Netherlands
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1 INTRODUCTION

This contribution describes a new approach to numerically solve the problem of finding optimal routes in continuous time and space, for instance in *modeling pedestrian route choice modelling* (see Hughes (2002), Hoogendoorn and Bovy (2005), and in *location choice theory* (see Beckman (1952), Puu and Beckman (1999), Yang and Wong (2000)).

This contribution focuses on the numerical solution of the problem. The numerical solution approach proposed by Hoogendoorn and Bovy (2004) entails using an equidistant, rectangular grid. This method yields problems in particular when considering areas that have a complex shape, or in which complex obstacles are present. Using an equidistant grid then often requires one to use a fine-grained mesh, which may yield computational difficulties in terms of computation time and memory use. For many applications, a non-equidistant mesh will be beneficial since this will allow use a fine-grain mesh, only where necessary. Finite element approaches have been put forward to solve the static shortest path problem in continuous space (Ho and Wong,2005). These approaches use triangulatization to approximate the area modeled. However, the approach is quite complex and computationally demanding.

The approach put forward in this contribution is more straightforward and based on the analogy with controlled Markov jump processes. It is computationally efficient, while maintaining the flexibility to use a non-equidistant triangular mesh.

2 PROBLEM FORMULATION

Infrastructure is described by an area $\Omega \subset I\!R^2$ in which the travelers move. The travelers enter the infrastructure at the origin areas $O_i \subset \Omega$, and leave via the destination areas $D_j \subset \Omega$. Both the origin and destination areas are described by closed sets. Note that travelers may use any point in the destination area to exit the facility. We assume that the time the traveler enters the facility is fixed. Within the infrastructure, obstacles $B_m \subset \Omega$ may be present. These obstacles reflect physical obstructions for the travelers, which is the reason why travelers will have to move around them while traveling to their destination.

2.1 Trajectories and velocity paths

A *feasible trajectory* is any possible movement through continuous time and space, mathematically defined by a parameterized curve

$$\vec{x}_{[t,T)} = \left\{ \vec{x}(s) \in \Omega \mid t \leq s \leq T, \vec{x}(s) \notin B_m \right\} \tag{1}$$

where t denotes the departure time and T denotes the terminal time. Considering a traveler going from origin $O_i \subset \Omega$ to destination $D_j \subset \Omega$, we would have $\vec{x}(t) \in O_i$ if t is the departure time. The final position $\vec{x}(T)$ may either be in the destination area D_j or not. In the former case, T is the arrival time of the traveler at the destination area. In the latter case, T is the end of the planning period, or the time the destination area is not available anymore (e.g. depature time of a train).

At this point, we note that the approach described in this article is *destination-oriented*, which implies that behavior of travelers can be described by the location $x(t)$ and their destination, and is thus independent on the origin. Furthermore, sub-paths of optimal paths will turn out to be optimal as well. In solving the problem, we generally consider the optimal path of a traveler that has (somehow) arrived at some location $\vec{x}(t')$ for $t' \geq t$ to a destination D_j.

Rather than the trajectories, the velocities $\vec{v}_{[t,T)}$ along the trajectories will be used as the main decision variable of the travelers, for mathematical convenience only. These *velocity trajectories* are defined by

$$\vec{v}_{[t,T)} = \left\{ \vec{v}(s) \in \Gamma \mid t \leq s < T \right\} \tag{2}$$

where Γ denotes the set of admissible velocities. Note that the *velocity* $v = eV \in I\!R^2$ of a traffic unit in the continuous case describes both its *speed* $V \in I\!R$ as well as its *(unit) direction* $\vec{e} \in I\!R^2, |\vec{e}| = 1$. The set of admissible velocities describes the constraints both caused by the infrastructure (travelers cannot walk into an obstacles, or in the direction opposite in the moving direction of an escalator), and by the flow conditions (traveler speed less or equal to a density-dependent speed limit).

For the path choice modeling in continuous time and space, let us recall the approach from Hoogendoorn and Bovy (2004). The key to the approach is that *instead* of determining the optimal paths $\vec{x}^*_{[t,T)}$ of travelers with destination D_j explicitly, only the local choice behavior of the travelers is considered. That is, consider a traveler who is located at x at instant t and is traveling towards destination D_j. The dynamic programming approach from Hoogendoorn and Bovy (2004) will describe the optimal velocity $\vec{v}^*_j(t,x)$ that the traveler needs to apply to reach the destination D_j as cheaply as possible.

2.2 Mathematical problem formulation

The optimal velocity $\vec{v}^*(t,\vec{x})$ of a traveler moving in a two-dimensional area Ω is a function of the *minimum travel cost field* $W(t,\vec{x})$ of traveling from location $\vec{x}$ at instant t and reaching the destination D before instant T. In turn, $W(t,\vec{x})$ can be determined by solving the so-called *Hamilton-Jacobi-Bellman* (HJB) equation; see (Hoogendoorn and Bovy,2005) for details):

$$-\frac{\partial}{\partial t} W(t,\vec{x}) = H\left(t,\vec{x},\nabla W\right) \tag{3}$$

with *terminal conditions* reflecting the penalty of not arriving at D before the end-time t_1

$$W(t_1,\vec{x}) = \phi_0 \tag{4}$$

and *boundary conditions* describing the utility of arriving at the destination area D_j at time T

$$W(T,\vec{x}) = -U(T) \text{ for } \vec{x} \in D \text{ and } T < t_1 \tag{5}$$

The so-called *Hamilton function* H is defined by

$$H\left(t,\vec{x},\nabla W\right) = \min_{\vec{v} \in \Gamma(\vec{x})} \left\{ L(t,\vec{x},\vec{v}) + \vec{v} \cdot \nabla W \right\} \tag{6}$$

where $\vec{v} \in I\!R^2$ denotes the velocity, and where L denotes the so-called running cost; $\Gamma(\vec{x})$ denotes the set of admissible velocities (i.e. admissible direction and speed, given prevailing flow conditions, obstacles) at instant t and location $\vec{x}$. The set will among other things comprise the fact that the *speed* $V = \|\vec{v}\|$ will in general be limited to some maximum speed V_{max}.

The *optimal velocity* $\vec{v}^*(t,\vec{x})$ chosen at instant *t* and location $\vec{x}$ can be expressed by the following relation:

$$\vec{v}^*(t,\vec{x}) = \arg\min\left\{L(t,\vec{x},\vec{v}) + \vec{v}\cdot\nabla W \mid \text{subject to } \vec{v}\in\Gamma(\vec{x})\right\} \tag{7}$$

The optimal path $\vec{x}^*_{[t,T)}$ can then be determined by integration of the optimal speeds, i.e.

$$\vec{x}^*(t') = \int_t^{t'} \vec{v}^*\left(s,\vec{x}^*(s)\right)ds \tag{8}$$

3 NUMERICAL SOLUTION APPROACH

For practical application of equations describing route choice in continuous time and space, we have proposed a numerical solution algorithm (Hoogendoorn and Bovy,2005). The approach applies the concepts proposed in (Fleming and Soner,1993) to the dynamic route choice problem in continuous space. The proposed approach is very straightforward, but it has a number of disadvantages. In particular, the fact that the approach requires *an equidistant grid* yields problems when considering problem areas of realistic size. This holds in particular when for practical applications where there is a (local) need for a fine grid due to the complexity of the infrastructure or (small) obstacles that are to be modeled. Especially under these circumstances, there is a need for approaches using non-equidistant grids, e.g. by triangulation.

3.1 Approximate solution approach using triangular mesh

In the first step of the approach, a triangular mesh is made of the area, including its obstacles and destination areas. A good approach to do this is using Delauny triangularization approach. Figure 1 shows an example of such as mesh for a complex area. Alternative meshes are possible as well. A detailed discussion on the different techniques that can be used for mesh generation is out of the scope of this paper.

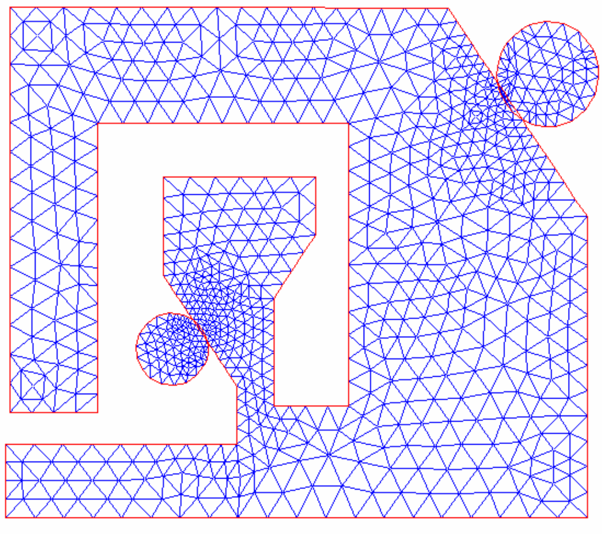

Figure 1 Example of a triangular mesh for a complex walking area.

The mesh can generally be described by corner points or *nodes* $\vec{x}$ which are connected by the edges or *links*. Let $\Psi(\vec{x})$ denote the set of neighboring nodes $\vec{y}$ of $\vec{x}$, i.e. nodes in the direct vicinity. Suppose that at time instant *t+h*, the minimum cost *W* of getting to the destination area *D* is known. Note that at the terminal time *T*, we precisely know the costs for all $\vec{x}\in\Omega$.

3.2 Equivalence with stochastic Markov jump processes

Let $\Psi(\vec{x})$ denote the set of nodes $\vec{y}$ connected to node $\vec{x}$. Soner and Fleming (1993) show that, upon applying the velocity $\vec{v}$, the probability of jumping from node $\vec{x}$ to node $\vec{y}$ during a period of length h (the timestep) is equal to:

$$p^{\vec{v}}(\vec{x},\vec{y}) = \frac{h}{\delta(\vec{x},\vec{y})}\left(\vec{v}\cdot\vec{e}(\vec{x},\vec{y})\right)^{+} \quad \text{where} \quad \delta(\vec{x},\vec{y}) = \left\|\vec{y}-\vec{x}\right\| \quad \text{and} \quad \vec{e}(\vec{x},\vec{y}) = \frac{\vec{y}-\vec{x}}{\delta} \tag{9}$$

In this notation, $(a)^{+} = \max(0,a)$. This implies that if the velocity $\vec{v}$ has a zero or negative component in the direction $\vec{y}-\vec{x}$, the probability of jumping from node $\vec{x}$ to node $\vec{y}$ is zero.

For the transition probability of staying in node $\vec{x}$ during the interval $[t,t+h]$ we have:

$$p^{\vec{v}}(\vec{x},\vec{x}) = 1-\sum_{\vec{y}\in\Psi(\vec{x})}\frac{h}{\delta(\vec{x},\vec{y})}\left(\vec{v}\cdot\vec{e}(\vec{x},\vec{y})\right)^{+} = 1-\sum_{\vec{y}\in\Psi(\vec{x})} p^{\vec{v}}(\vec{x},\vec{y}) \tag{10}$$

We can now show that the expected cost incurred when applying velocity $\vec{v}$ – given that from that point onwards the *optimal velocities* are applied – equals:

$$h\cdot L(t,\vec{x},\vec{v}) + \sum_{\vec{y}\in\Psi(\vec{x})\cup\vec{x}} p^{\vec{v}}(\vec{x},\vec{y})\cdot W(t+h,\vec{y}) \tag{11}$$

Consequently, we can compute the value function (or minimum expected cost function) easily by:

$$W(t,\vec{x}) = \min_{\vec{v}\in\Gamma(\vec{x})}\left[h\cdot L(t,\vec{x},\vec{v}) + \sum_{\vec{y}\in\Psi(\vec{x})\cup\vec{x}} p^{\vec{v}}(\vec{x},\vec{y})\cdot W(t+h,\vec{y})\right] \tag{12}$$

This equation can be used directly to numerically solve the route choice problem in continuous time and space. This is done backwards in time: starting from the terminal conditions, we can apply Eq. (1.4) to solve the problem for times $t = T\text{-}h$, $T\text{-}2h,\ldots,t_0$.

4 APPLICATION EXAMPLE

Let us illustrate application of the example shown in Figure 1. Consider the situation in which the travelers aim to get from any location to the position in the left lower corner. Only travel time is considered, i.e. $L = 1$ (this implies that there is no penalty of moving close to obstacles or borders). Furthermore, for the terminal cost we will use:

$$W(t_1,\vec{x}) = \infty \tag{13}$$

For the boundary conditions we have:

$$W(T,\vec{x}) = 0 \text{ for } \vec{x}\in D \text{ and } T < t_1 \tag{14}$$

The maximum speed is set equal to 1.

Figure 2 shows the resulting minimum cost of getting to the destination D. The route choice can then be easily determined from the fact that travelers will move into the direction in which the costs decline most rapidly.

Note that a finite value of the minimum cost is computed for each location in the area. This implies that from each location, the destination can be reached in time. Figure 3 shows the project of the solution at time instant $t = t_1 - 25$ (i.e. 25 seconds before the terminal time). For a large number of nodes, the minimum costs is equal to ∞, implying that the destination *cannot be reached before the end-time* t_1.

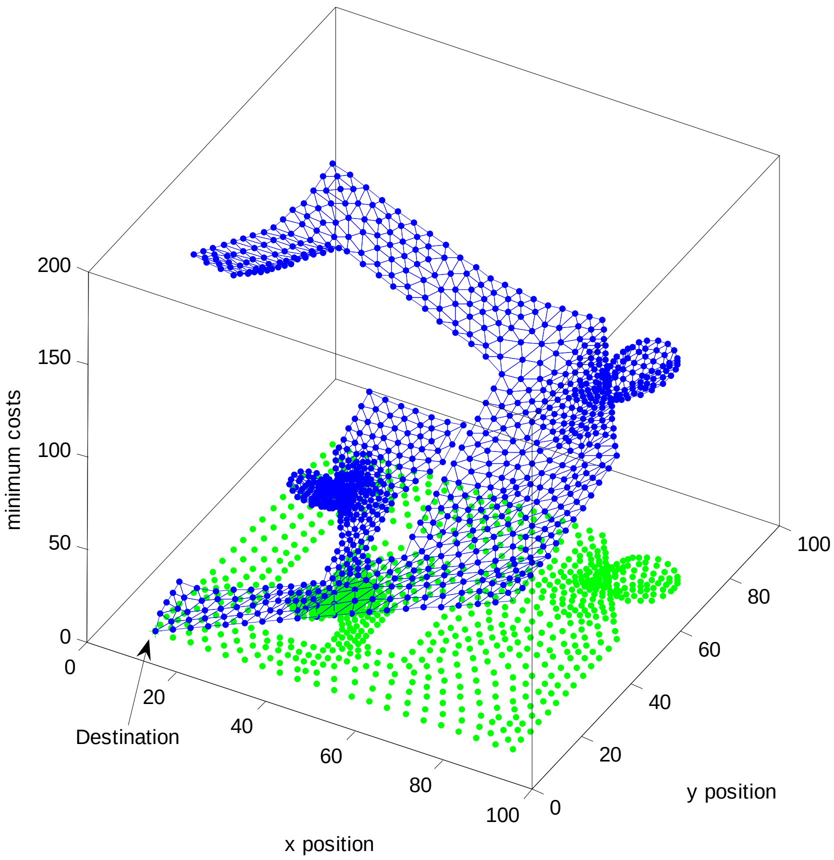

Figure 2 Optimal costs of getting towards destination at time t = 0. Note that the destination can be reached within time from all possible locations.

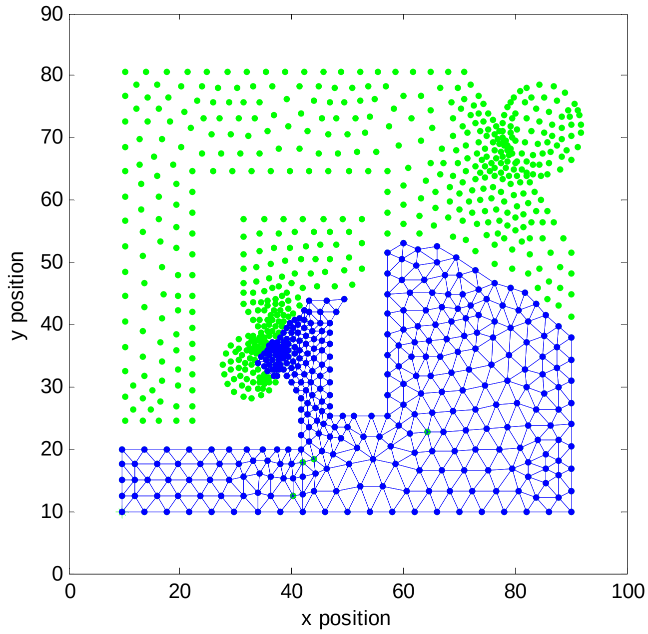

Figure 3 Locations from which the destination *D* is reachable at time $t_1 - 25$.

5 CONCLUSIONS

In this contribution, a new approach was put forward for numerically solving the route choice problem in continuous time and space. The approach allows using non-equidistant meshes, while maintaining computational efficiency and being relatively straightforward. The approach is illustrated by means of a numerical example where travelers have to choice their route through a complex shaped area.

Future research will be focused on addressing the accuracy of the approach and its relation with the size of the triangles. Furthermore, the approach will be implemented in the pedestrian flow simulation model NOMAD (Hoogendoorn and Bovy,2003).

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