New Methods to Compute Dynamic Bid-Prices in Network Revenue Management

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Bid-prices form a powerful tool for constructing good policies for network revenue management problems. The fundamental idea is to associate a bid-price with each flight leg, which captures the opportunity cost of a unit of capacity. An itinerary request is accepted only when the revenue from the requested itinerary exceeds the sum of the bid-prices of the flight legs in the requested itinerary; see Williamson (1992), Talluri and van Ryzin (1998) and Talluri and van Ryzin (2004).

Bid-prices are traditionally computed by solving a deterministic linear program. However, this linear program tends to be somewhat crude in the sense that it uses only the expected numbers of the itinerary requests that are to arrive until the time of departure and does not incorporate the probability distributions or temporal dynamics of the arrivals of the itinerary requests. In practice, as the itinerary requests arrive over time, the deterministic linear program is periodically resolved to *artificially* incorporate the temporal dynamics of the arrivals of the itinerary requests.

In this paper, we present two new methods for computing bid-prices. Both of these methods partially incorporate the temporal dynamics of the arrivals of the itinerary requests. The fundamental idea is to formulate the network revenue management problem as a dynamic program and to relax certain constraints by associating Lagrange multipliers with them. As a result, the network revenue management problem decomposes by the flight legs and we can concentrate on one flight leg at a time. The methods that we present naturally yield upper bounds on the maximum expected revenue over the planning horizon, remain applicable in the presence of cancellations, and provide a new and refined deterministic linear program for the network revenue management problem.

1 Problem Formulation

We have a set of flight legs that can be used to satisfy the itinerary requests that arrive randomly over time. Whenever an itinerary request arrives, we have to decide whether to accept or reject it. An accepted itinerary request generates a revenue and consumes the capacities on the relevant flight legs. A rejected itinerary request simply leaves the system.

The problem takes place over the finite planning horizon $\mathcal{T} = \{1, \dots, \tau\}$ and all flight legs depart at time period $\tau + 1$. The set of flight legs is $\mathcal{L}$ and the set of itineraries is $\mathcal{J}$. The capacity on flight leg i is c_i. If a request for itinerary j is accepted, then we generate a revenue of f_j and consume a_{ij} units of capacity on flight leg i. If flight leg i is not in itinerary j, then we have $a_{ij} = 0$. The probability that a request for itinerary j arrives at time period t is p_{jt}. For notational brevity, we assume that $\sum_{j \in \mathcal{J}} p_{jt} = 1$. If there is a positive probability that no itinerary requests arrive at time period t, then we can cover this case by defining a fictitious itinerary ϕ with $f_\phi = 0$ and $p_{\phi t} = 1 - \sum_{j \in \mathcal{J}} p_{jt}$.

We let x_{it} be the remaining capacity on flight leg i at time period t so that $x_t = \{x_{it} : i \in \mathcal{L}\}$ gives the state of the system at time period t. We capture the decisions at time period t by $y_t = \{y_{jt} : j \in \mathcal{J}\}$, where y_{jt} takes value 1 if a request for itinerary j is accepted at time period t, and 0 otherwise. Letting e_i be the $|\mathcal{L}|$-dimensional unit vector with a 1 in the element corresponding to $i \in \mathcal{L}$, the optimal policy can be found by computing the value functions $\{V_t(\cdot) : t \in \mathcal{T}\}$ through the optimality equation

$$\begin{aligned} V_t(x_t) = \max \quad & \sum_{j \in \mathcal{J}} p_{jt} \Big[f_j \, y_{jt} + V_{t+1}(x_t - y_{jt} \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, e_i) \Big] && && (1) \\ \text{subject to} \quad & a_{ij} \, y_{jt} \le x_{it} && \text{for all } i \in \mathcal{L}, \; j \in \mathcal{J} && (2) \\ & y_{jt} \in \{0, 1\} && \text{for all } j \in \mathcal{J}. && (3) \end{aligned}$$

Given the state variable x_t, it is easy to see that the optimal decisions at time period t are given by $\hat{y}_t(x_t) = \{\hat{y}_{jt}(x_t) : j \in \mathcal{J}\}$, where

$$\hat{y}_{jt}(x_t) = \begin{cases} 1 & \text{if } f_j + V_{t+1}(x_t - \sum_{i \in \mathcal{L}} a_{ij} \, e_i) \ge V_{t+1}(x_t) \text{ and } a_{ij} \le x_{it} \text{ for all } i \in \mathcal{L} \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

2 Capacity-Based Lagrangian Relaxation

Associating the positive Lagrange multipliers $\lambda = \{\lambda_{ijt} : i \in \mathcal{L}, \; j \in \mathcal{J}, \; t \in \mathcal{T}\}$ with constraints (2), we propose solving the optimality equation

$$\hat{V}_t(x_t \,|\, \lambda) = \max_{y_t \in \{0,1\}^{|\mathcal{J}|}} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \left\{ \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \lambda_{ijt} \Big] y_{jt} + \sum_{i \in \mathcal{L}} \lambda_{ijt} \, x_{it} + \hat{V}_{t+1}(x_t - y_{jt} \textstyle\sum_{i \in \mathcal{L}} a_{ij} \, e_i \,|\, \lambda) \right\} \right\},$$

where we scale the Lagrange multipliers by $\{p_{jt} : j \in \mathcal{J}\}$ for notational brevity and use the argument λ in the value functions to emphasize that the solution to the optimality equation above depends on the Lagrange multipliers. The next proposition shows that there is a simple solution to this optimality equation. In the next proposition and throughout the rest of the paper, we let $r^{\lambda}_{it} = \sum_{j \in \mathcal{J}} p_{jt} \, \lambda_{ijt} + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} \, \lambda_{ij\tau}$ with the boundary condition that $r^{\lambda}_{i,\tau+1} = 0$.

Proposition 1 *Letting* $[\cdot]^+ = \max\{0, \cdot\}$, *we have*

$$\begin{aligned} \hat{V}_t(x_t \,|\, \lambda) = \sum_{i \in \mathcal{L}} r^{\lambda}_{it} \, x_{it} + \sum_{j \in \mathcal{J}} p_{jt} \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \lambda_{ijt} - \sum_{i \in \mathcal{L}} a_{ij} \, r^{\lambda}_{i,t+1} \Big]^+ \\ + \ldots + \sum_{j \in \mathcal{J}} p_{j\tau} \Big[f_j - \sum_{i \in \mathcal{L}} a_{ij} \, \lambda_{ij\tau} - \sum_{i \in \mathcal{L}} a_{ij} \, r^{\lambda}_{i,\tau+1} \Big]^+. \end{aligned} \quad (5)$$

The next proposition shows that we obtain an upper bound on the value function by using this Lagrangian relaxation strategy.

Proposition 2 *If the Lagrange multipliers are positive, then we have* $V_t(x_t) \le \hat{V}_t(x_t \,|\, \lambda)$.

Since the initial leg capacities are given by $c = \{c_i : i \in \mathcal{L}\}$, the maximum expected revenue over the planning horizon is $V_1(c)$ and Proposition 2 implies that we can obtain the tightest possible upper bound on $V_1(c)$ by solving the problem $\min_{\lambda \geq 0} \{\hat{V}_1(c \,|\, \lambda)\}$. It is clear from (5) that the objective function of this problem is a convex function of λ.

An alternative method to find good policies for the network revenue management problem is to use a deterministic linear program. Letting w_j be the number of requests for itinerary j that we plan to accept over the planning horizon, this linear program has the form

$$\max \quad \sum_{j \in \mathcal{J}} f_j \, w_j \tag{6}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} a_{ij} \, w_j \leq c_i \qquad \text{for all } i \in \mathcal{L} \tag{7}$$

$$0 \leq w_j \leq \sum_{t \in \mathcal{T}} p_{jt} \qquad \text{for all } j \in \mathcal{J}; \tag{8}$$

see Talluri and van Ryzin (2004). There are two uses of problem (6)-(8). First, letting $\{\hat{\mu}_i : i \in \mathcal{L}\}$ be the optimal values of the dual variables associated constraints (7), we can use $\hat{\mu}_i$ as an estimate of the opportunity cost of a unit of capacity on flight leg i. These opportunity costs are referred to as the bid-prices in the network revenue management vocabulary and they are often used to decide whether to accept or reject an itinerary request. The decision rule is that if the revenue from an itinerary request exceeds the sum of the bid-prices of the flight legs in the requested itinerary, then we accept the itinerary request subject to the capacity availability. Specifically, if we have

$$f_j \geq \sum_{i \in \mathcal{L}} a_{ij} \, \hat{\mu}_i \tag{9}$$

and $a_{ij} \leq x_{it}$ for all $i \in \mathcal{L}$, then we accept a request for itinerary j. Second, one can show that the optimal objective value of problem (6)-(8) provides an upper bound on the maximum expected revenue over the planning horizon. The next proposition shows that we obtain a tighter upper bound by solving the problem $\min_{\lambda \geq 0} \{\hat{V}_1(c \,|\, \lambda)\}$ than by solving problem (6)-(8).

Proposition 3 *If we use $\hat{\zeta}$ to denote the optimal objective value of problem* (6)-(8)*, then we have* $V_1(c) \leq \min_{\lambda \geq 0} \{\hat{V}_1(c \,|\, \lambda)\} \leq \hat{\zeta}$.

Interestingly, the next proposition shows that solving the problem $\min_{\lambda \geq 0} \{\hat{V}_1(c \,|\, \lambda)\}$ is equivalent to solving a deterministic linear program similar to problem (6)-(8).

Proposition 4 *The optimal objective value of the problem* $\min_{\lambda \geq 0} \{\hat{V}_1(c \,|\, \lambda)\}$ *is the same as that of*

$$\max \quad \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} f_j \, w_{jt} \tag{10}$$

$$\text{subject to} \quad \sum_{j \in \mathcal{J}} p_{jt} \, a_{ij} \, w_{j1} + \ldots + \sum_{j \in \mathcal{J}} p_{jt} \, a_{ij} \, w_{j,t-1} + a_{ij} \, w_{jt} \leq p_{jt} \, c_i \quad \text{for all } i \in \mathcal{L}, \ j \in \mathcal{J}, \ t \in \mathcal{T} \tag{11}$$

$$0 \leq w_{jt} \leq p_{jt} \qquad \text{for all } j \in \mathcal{J}, \ t \in \mathcal{T}. \tag{12}$$

Therefore, Propositions 3 and 4 show that one can formulate a deterministic linear program that yields a tighter upper bound than does problem (6)-(8). Finally, we note that Talluri and van Ryzin (1998) show the asymptotic optimality of the decision rule in (9) as the capacities on the flight legs and the expected numbers of itinerary requests increase linearly with the same rate. It is possible to show that the same asymptotic optimality result holds when one uses the optimal values of the dual variables associated with constraints (11) in problem (10)-(12) as the bid-prices.

The Lagrangian relaxation strategy that we present in this section can easily be extended to network revenue management problems with cancellations. We only present one representative result here. In the next proposition, for notational brevity, we assume that the cancellations are not refunded. We also assume that the cancellations of different reservations and at different time periods are independent. Finally, we let Q_{jt} be the probability that a reservation for itinerary j at time period t is *retained* until the time of departure and b_j be the penalty of not honoring a reservation for itinerary j.

Proposition 5 *The optimal objective value of the problem*

$$\begin{aligned}
\max \quad & \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}} f_j\, u_{jt} - \sum_{j\in\mathcal{J}} b_j\, v_j && && (13)\\
\textit{subject to} \quad & \sum_{t\in\mathcal{T}}\sum_{j\in\mathcal{J}} a_{ij}\, Q_{jt}\, u_{jt} - \sum_{j\in\mathcal{J}} a_{ij}\, v_j \le c_i && \textit{for all } i\in\mathcal{L} && (14)\\
& \sum_{t\in\mathcal{T}} Q_{jt}\, u_{jt} - v_j \ge 0 && \textit{for all } j\in\mathcal{J} && (15)\\
& 0 \le u_{jt} \le p_{jt} && \textit{for all } j\in\mathcal{J},\ t\in\mathcal{T} && (16)\\
& v_j \ge 0 && \textit{for all } j\in\mathcal{J} && (17)
\end{aligned}$$

provides an upper bound on the maximum expected revenue over the planning horizon.

3 Leg-Based Lagrangian Relaxation

We begin by introducing some new notation. We augment $\mathcal{L}$ by a fictitious flight leg ψ with infinite capacity. We extend the decisions at time period t as $y_t = \{y_{ijt} : i \in \mathcal{L}\cup\{\psi\},\ j \in \mathcal{J}\}$, where y_{ijt} takes value 1 if we accept flight leg i when a request for itinerary j arrives at time period t, and 0 otherwise. In this case, it is easy to see that the optimality equation

$$\begin{aligned}
V_t(x_t) = \max \quad & \sum_{j\in\mathcal{J}} p_{jt} \Big\{ f_j\, y_{\psi jt} + V_{t+1}(x_t - \sum_{i\in\mathcal{L}} y_{ijt}\, a_{ij}\, e_i) \Big\} && && (18)\\
\text{subject to} \quad & a_{ij}\, y_{ijt} \le x_{it} && \text{for all } i\in\mathcal{L},\ j\in\mathcal{J} && (19)\\
& y_{ijt} - y_{\psi jt} = 0 && \text{for all } i\in\mathcal{L},\ j\in\mathcal{J} && (20)\\
& y_{ijt} \in \{0,1\} && \text{for all } i\in\mathcal{L}\cup\{\psi\},\ j\in\mathcal{J} && (21)
\end{aligned}$$

is equivalent to the optimality equation in (1). Since the capacity on the fictitious flight leg is infinite, we do not keep track of it in our state variable and the state variable in the dynamic program above is still $x_t = \{x_{it} : i \in \mathcal{L}\}$. In the feasible solution set of problem (18)-(21), only constraints (20) link the different flight legs. This suggests associating the Lagrange multipliers $\alpha = \{\alpha_{ijt} : i \in \mathcal{L},\ j \in \mathcal{J},\ t \in \mathcal{T}\}$ with these constraints and solving the dynamic program

$$\tilde{V}_t(x_t \,|\, \alpha) = \max \quad \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \big[f_j - \sum_{i \in \mathcal{L}} \alpha_{ijt} \big]\, y_{\psi jt} + \sum_{i \in \mathcal{L}} \alpha_{ijt}\, y_{ijt} + \tilde{V}_{t+1}(x_t - \sum_{i \in \mathcal{L}} y_{ijt}\, a_{ij}\, e_i \,|\, \alpha) \Big\} \tag{22}$$

$$\text{subject to} \quad (19), (21), \tag{23}$$

where we again scale the Lagrange multipliers by $\{p_{jt} : j \in \mathcal{J}\}$ for notational clarity. Letting $y_{it} = \{y_{ijt} : j \in \mathcal{J}\}$, we define the set $\mathcal{Y}_{it}(x_{it}) = \{y_{it} \in \{0,1\}^{|\mathcal{J}|} : a_{ij}\, y_{ijt} \le x_{it} \text{ for all } j \in \mathcal{J}\}$, in which case constraints (19) and (21) can succinctly be written as $y_{it} \in \mathcal{Y}_{it}(x_{it})$ for all $i \in \mathcal{L}$ and $y_{\psi t} \in \{0,1\}^{|\mathcal{L}|}$. The following proposition shows that the optimality equation in (22)-(23) decomposes by the flight legs.

Proposition 6 *If $\{\vartheta_{it}(\cdot \,|\, \alpha) : t \in \mathcal{T}\}$ is the solution to the optimality equation*

$$\vartheta_{it}(x_{it} \,|\, \alpha) = \max_{y_{it} \in \mathcal{Y}_{it}(x_{it})} \left\{ \sum_{j \in \mathcal{J}} p_{jt} \Big\{ \alpha_{ijt}\, y_{ijt} + \vartheta_{i,t+1}(x_{it} - a_{ij}\, y_{ijt} \,|\, \alpha) \Big\} \right\} \tag{24}$$

for all $i \in \mathcal{L}$, then we have $\tilde{V}_t(x_t \,|\, \alpha) = \sum_{t'=t}^{\tau} \sum_{j \in \mathcal{J}} p_{jt'} \big[f_j - \sum_{i \in \mathcal{L}} \alpha_{ijt'} \big]^+ + \sum_{i \in \mathcal{L}} \vartheta_{it}(x_{it} \,|\, \alpha)$.

Therefore, we can efficiently solve the optimality equation in (22)-(23) by concentrating on one flight leg at a time. Results similar to Propositions 2 and 3 can be established for the optimality equation in (22)-(23). In particular, we have $V_t(x_t) \le \tilde{V}_t(x_t \,|\, \alpha)$ and $V_1(c) \le \min_\alpha \{\tilde{V}_1(c \,|\, \alpha)\} \le \hat{\zeta}$. Consequently, we can solve the problem $\min_\alpha \{\tilde{V}_1(c \,|\, \alpha)\}$ to obtain the tightest possible upper bound on $V_1(c)$.

4 Computational Experiments

In this section, we compare the performances of the two Lagrangian relaxation strategies presented in Sections 2 and 3 with the performance of the bid-prices obtained by solving problem (6)-(8). When applying the Lagrangian relaxation strategy in Section 2 in practice, given the state variable x_t at time period t, we solve the problem $\min_{\lambda \ge 0} \{\hat{V}_t(x_t \,|\, \lambda)\}$ to obtain the optimal solution $\hat{\lambda}(t, x_t)$. In this case, we make the decisions at time period t by replacing the value function $V_{t+1}(\cdot)$ in (4) with $\hat{V}_{t+1}(\cdot \,|\, \hat{\lambda}(t, x_t))$. We refer to this solution method as CP-R. We use a similar approach when applying the Lagrangian relaxation strategy in Section 3 in practice. In particular, we solve the problem $\min_\alpha \{\tilde{V}_t(x_t \,|\, \alpha)\}$ to compute the Lagrange multipliers at each time period. We refer to this solution method as LG-R. Finally, when using problem (6)-(8) to compute bid-prices in practice, given the state variable x_t at time period t, we replace the right sides of constraints (7) with $\{x_{it} : i \in \mathcal{I}\}$ and the right sides of constraints (8) with $\{\sum_{t'=t}^{\tau} p_{jt'} : j \in \mathcal{J}\}$, and solve this problem to obtain the optimal values $\{\hat{\mu}_i(t, x_t) : i \in \mathcal{I}\}$ of the dual variables associated with constraints (7). We use $\{\hat{\mu}_i(t, x_t) : i \in \mathcal{I}\}$ as the bid-prices in (9). We refer to this solution method as LR.

Table 1 summarizes our computational results. The first column in this table shows the characteristics of the test problems. For the four-tuple (x_1, x_2, x_3, x_4) in this column, x_1 represents the number of flight legs, x_2 represents the number of itineraries and x_3 represents the ratio of the total expected

Problem	$\hat{V}_1(c \mid \lambda^*)$	$\tilde{V}_1(c \mid \alpha^*)$	$\hat{\zeta}$	$\frac{\hat{V}_1(c \mid \lambda^*)}{\hat{\zeta}}$	$\frac{\tilde{V}_1(c \mid \alpha^*)}{\hat{\zeta}}$	CP-R	LG-R	LP	$\frac{\text{CP-R}}{\text{LP}}$	$\frac{\text{LG-R}}{\text{LP}}$
$(6, 24, 1.0, 2)$	5,864	5,538	5,966	98.29	92.83	5,460	5,471	5,456	100.07	100.27
$(6, 24, 1.0, 4)$	8,360	7,945	8,478	98.61	93.71	7,829	7,896	7,718	101.44	102.31
$(6, 24, 1.0, 8)$	13,383	12,878	13,501	99.13	95.39	12,695	12,871	12,248	103.65	105.09
$(8, 40, 1.0, 2)$	7,320	6,848	7,460	98.12	91.80	6,670	6,696	6,659	100.17	100.56
$(8, 40, 1.0, 4)$	10,521	9,908	10,691	98.41	92.68	9,566	9,654	9,362	102.18	103.12
$(8, 40, 1.0, 8)$	16,978	16,228	17,152	98.99	94.61	15,496	15,873	14,900	104.00	106.53
$(6, 24, 1.2, 2)$	5,231	4,984	5,339	97.98	93.35	4,860	4,889	4,833	100.56	101.16
$(6, 24, 1.2, 4)$	7,727	7,356	7,851	98.42	93.70	7,174	7,272	6,950	103.22	104.63
$(6, 24, 1.2, 8)$	12,750	12,255	12,874	99.04	95.19	11,855	12,227	11,164	106.19	109.52
$(8, 40, 1.2, 2)$	6,509	6,167	6,691	97.28	92.17	5,867	5,941	5,861	100.10	101.36
$(8, 40, 1.2, 4)$	9,702	9,175	9,921	97.79	92.48	8,681	8,875	8,410	103.22	105.53
$(8, 40, 1.2, 8)$	16,156	15,478	16,382	98.62	94.48	14,556	15,047	13,574	107.23	110.85
$(6, 24, 1.6, 2)$	4,367	4,150	4,483	97.41	92.57	4,014	4,060	3,994	100.50	101.65
$(6, 24, 1.6, 4)$	6,857	6,470	6,995	98.03	92.49	6,269	6,378	6,037	103.84	105.65
$(6, 24, 1.6, 8)$	11,880	11,343	12,018	98.85	94.38	10,818	11,268	10,093	107.18	111.64
$(8, 40, 1.6, 2)$	5,216	4,943	5,401	96.57	91.52	4,680	4,699	4,640	100.86	101.27
$(8, 40, 1.6, 4)$	8,406	7,905	8,632	97.38	91.58	7,312	7,482	7,032	103.98	106.40
$(8, 40, 1.6, 8)$	14,858	14,165	15,093	98.44	93.85	12,721	13,483	11,817	107.65	114.10

Table 1: Summary of computational results.

demand to the total available capacity. For each origin-destination pair, there is a high-fare and a low-fare itinerary, and the high-fare itineraries are x_4 times more expensive than the low-fare itineraries. All of our test problems involve a hub-and-spoke network with one hub. Letting λ^* and α^* be the optimal solutions to the problems $\min_{\lambda \geq 0} \{\hat{V}_1(c \mid \lambda)\}$ and $\min_{\alpha} \{\tilde{V}_1(c \mid \alpha)\}$, the second, third and fourth columns show $\hat{V}_1(c \mid \lambda^*)$, $\tilde{V}_1(c \mid \alpha^*)$ and $\hat{\zeta}$. Finally, the seventh, eight and ninth columns show the expected revenues obtained over the planning horizon by CP-R, LG-R and LP. These expected revenues are estimated through simulation.

The results indicate that the upper bounds obtained by LG-R are always tighter than the lower bounds obtained by CP-R. The gap between the upper bounds obtained by LG-R and LP can be as high as 8.5%. The expected revenues obtained by CP-R and LG-R are always better than the ones obtained by LP. The performance of LG-R is always better than the performance of CP-R. The performance gap between LG-R and LP can be as high as 14 %. The performance gap between LG-R and LP seems to increase as the fare difference between the high-fare and low-fare itineraries increases and as the gap between the total expected demand and the total available capacity increases.

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