A CONTINUOUSLY DIFFERENTIABLE OPTIMIZATION APPROACH FOR THE EQUILIBRIUM-BASED ORIGIN-DESTINATION MATRIX ESTIMATION PROBLEM

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1. INTRODUCTION

With increasing land use resulting from social-economic development or/and change of transportation network infrastructure especially for developing countries, the origin-destination (O-D) matrix is frequently updated. Assuming that the route choice behavior of network users (drivers) follows the user equilibrium (UE) criterion, link traffic counts, i.e., observed link flows, are the result of UE traffic assignment problem with respect to an appropriate O-D matrix. It should be pointed out that these link traffic counts can be efficiently acquired by means of advanced traveler information systems. Hence, estimation of O-D matrix from link traffic counts, while taking into account the UE conditions, has already become a very interesting research topic in the past twenty years. For short, the problem is referred to as the UE-based O-D matrix estimation problem throughout this paper.

In the case of separable or symmetric link travel time functions, UE conditions possess a linearly constrained minimization model. The UE-based O-D matrix estimation problem can be formulated as a bilevel programming model. In the induced bilevel programming model, the upper level problem aims to minimize an error measure between observed and estimated link flows and the estimated O-D matrix and a target matrix, and the lower level problem is the convex minimization formulation for the symmetric UE traffic assignment problem. A fewer solution methods such as the sensitivity analysis based algorithm have been developed for solving the O-D matrix estimation problem. However, link travel time functions usually are asymmetric in practice. In the case of asymmetric link travel time functions, it is well recognized that the variational inequality (VI) can model the UE conditions. Therefore, the UE-based O-D matrix estimation can be modeled by the mathematical program with equilibrium constraints (MPEC) in which the VI formulation of the UE conditions has become a constraint of the model. To our best knowledge, few studies have been conducted for the asymmetric UE-based O-D matrix estimation.

This paper will first give the MPEC model for the asymmetric O-D matrix estimation problem for the congested networks. Applying the existing research results on the MPEC studies (e.g., Marcotte and Zhu, 1997), it can be easily demonstrated that the MPEC model can be converted into a single-level continuously differentiable minimization model by means of a continuously differentiable gap function derived for the strictly monotone VI by Fukushima (1992). Although

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the gap function does not have an explicit mathematical, we can rigorously show that its value and gradient can be evaluated by solving a special symmetric UE traffic assignment problem with the constructed linearly travel time functions. It should be pointed out the explicit expression of the gradient for the gap function is not a corollary of any result in the MPEC studies since VI formulation for the asymmetric UE conditions involves the interim variables, path flows. As for a solution method for the asymmetric O-D matrix estimation problem, we can employ the augmented Lagrangian algorithm incorporating with any asymmetric UE traffic assignment method. This is because the induced single level continuously differentiable minimization has a non-convex and nonlinear constraint that is alternative representation of the asymmetric UE conditions based on the gap function. The proposed model and algorithm will be evaluated by several numerical examples.

2. MATHEMATICAL MODELS

Let $G=(N,A)$ be a transportation network consisting of a set N of nodes and a set A of directed links. Each link $a\in A$ has an associated flow-dependent travel time function $t_a(v)$, where v is the vector of all link flows, $v=(\cdots,v_a,\cdots)^{\mathrm{T}}$ in which v_a is traffic flow on link $a\in A$. Let W denote the set of O-D pairs, q_w travel demand between O-D pair $w\in W$, P_w the set of all paths between O-D pair $w\in W$ and let f_r^w denote the traffic flow on path $r\in P_w$, $w\in W$ and f denote the vector of all path flows, i.e, $f=(\cdots,f_r^w,\cdots)^{\mathrm{T}}$. The topological relationship between any link $a\in A$ and any path $r\in P_w$, $w\in W$ is expressed by an indicator δ_{ar}^w, where $\delta_{ar}^w=1$ if path r between O-D pair $w\in W$ uses link a, and $\delta_{ar}^w=0$ otherwise. Let vector $\bar{v}=(\cdots,v_a,\cdots)^{\mathrm{T}}, a\in\bar{A}$, where $\bar{A}\subseteq A$ is the set of observed links and $\bar{v}_a$ is the observed traffic flow on link $a\in\bar{A}$, and $q=(\cdots,q_w,\cdots)^{\mathrm{T}}, w\in W$ the vector of all O-D demands which will be estimated from the observed link flow pattern $\bar{v}$. Given any O-D demand q, the set of all feasible link flows can be expressed by

$$\Omega(q)=\left\{v\,\middle|\,v_a=\sum_{w\in W}\sum_{r\in P_w} f_r^w\delta_{ar}^w, a\in A;\ \sum_{r\in P_w} f_r^w=q_w, w\in W;\ f_r^w\geq 0, r\in P_w, w\in W\right\} \tag{1}$$

It is assumed that link travel time function $t(v)$ is strictly monotone and continuously differentiable, and that the route choice behavior of network users follows the UE criterion. The user equilibrium-based O-D matrix estimation problem from observed link flows can be formulated as the MPEC as follows:

MPEC Model

$$\min_{v,q}\ F(v,q)=F_1(q,\bar{q})+F_2(v,\bar{v}) \tag{2}$$

subject to

$$q_w \geq 0, w \in W \tag{3}$$

$$v \in \Omega(q) \tag{4}$$

$$(\hat{v} - v)^{\mathrm{T}} t(v) \geq 0, \ \forall \hat{v} \in \Omega(q) \tag{5}$$

where vector $\bar{q}$ is the vector of all known target O-D demands, namely, $\bar{q} = (\cdots, \bar{q}_w, \cdots)^{\mathrm{T}}$ and $\bar{q}_w$ is the known target travel demand between O-D pair $w \in W$. Functions $F_1(q, \bar{q})$ and $\bar{F}_2(v, \bar{v})$ are two error measures between estimated O-D matrix and the target matrix and between estimated link flows and observed link flow.

In the above MPEC (2)-(5), the decision variables are link flow v and O-D demand q, and constraint (5) is a parameterized VI that describes the UE conditions. Although the MPEC model can perfectly characterizes nature of the UE-based O-D matrix estimation, the VI constraint (5) prevents direct application of the existing efficient solution methods developed in nonlinear programming.

Applying the VI gap function invented by Fukushima (1992), it can be seen that the parametric VI (5) is equivalent to solving the single equation:

$$G_\alpha(v, q) = 0 \tag{6}$$

where $G_\alpha(v, q)$ is the optimal value of the objective function for the following positive definite quadratic programming problem associated with two parameters q and v:

$$\min_{\hat{v}} \ H_\alpha(v, q, \hat{v}) = (\hat{v} - v)^{\mathrm{T}} t(v) + \frac{1}{2\alpha}(\hat{v} - v)^{\mathrm{T}} M(\hat{v} - v) \tag{7}$$

subject to

$$\hat{v}_a = \sum_{w \in W} \sum_{r \in P_w} f_r^w \delta_{ar}^w, \ a \in A \tag{8}$$

$$\sum_{r \in P_w} f_r^w = q_w, \ w \in W \tag{9}$$

$$f_r^w \geq 0, \ r \in P_w, \ w \in W \tag{10}$$

where α is any positive number and M can be any positive definite matrix. Alternative, function $G_\alpha(v, q)$ can be rewritten as

$$G_\alpha(v, q) = \min_{\hat{v} \in \Omega(q)} \ H_\alpha(v, q, \hat{v}) \tag{11}$$

By setting M as a diagonal matrix with positive diagonal elements $m_a, \ a \in A,$ the parameterized strictly convex minimization problem (7)-(10) becomes a special symmetric UE traffic assignment problem with parameters v and q:

$$\min_{\bar{v} \in \Omega(q)} \ H_\alpha(v, q, \hat{v}) = \sum_{a \in A} \int_0^{\hat{v}_a} \hat{t}_a(v, q, \omega) \mathrm{d}\omega \tag{12}$$

where $\hat{t}_a(v, y, \hat{v}_a)$, a linear travel time function with respect to $\bar{v}$, is of the form:

$$\hat{t}_a(v, q, \hat{v}_a) = t_a(v) + \frac{m_a}{\alpha}(\hat{v}_a - v_a), \quad a \in A \tag{13}$$

The following important property is a corollary of the theorems established by Fukushima (1992).

Proposition 1. *The gap function* $G_\alpha(v,q)$ *defined in eqn.* (11) *has the following properties*

(i) $G_\alpha(v,q) \le 0$ *;*

(ii) $G_\alpha(v,q) = 0$ *if and only if* v *is the equilibrium link flow pattern for given* y *;*

(iii) $G_\alpha(v,q)$ *is a continuously differentiable function with respect to* v *and* q

In fact, Proposition 1 implies that MPEC model (2)-(5) can be equivalently converted into the continuously differentiable minimization nonlinear model as follows.

Nonlinear Programming Model

$$\min_{v,q} F(v,q) \tag{14}$$

subject to

$$q_w \ge 0, w \in W \tag{15}$$

$$v \in \Omega(q) \tag{16}$$

$$G_\alpha(v,q) = 0 \tag{17}$$

By applying some convex analysis results, we can rigorously derive an explicit expression of the gradient for gap function $G_a(v,q)$, which is highlighted by Proposition 2.

Proposition 2. *At a pair of* (v,q), *gradient of gap function* $G_\alpha(v,q)$, $\nabla G_\alpha(v,q)$, *can be calculated by*

$$\frac{\partial G_\alpha(v,q)}{\partial q_w} = \hat{\mu}_w^*(v,q), w \in W \tag{18}$$

$$\frac{\partial G_\alpha(v,q)}{\partial v_a} = \sum_{b \in A} \left(\hat{v}_b^*(v,q) - v_b\right) \frac{\partial t_b(v,y)}{\partial v_a} - t_a(v,q) + \frac{m_a}{\alpha}\left(\hat{v}_a^*(v,q) - v_a\right), \; a \in A \tag{19}$$

where $\{\hat{v}_a^*(v,y), a \in A\}$ *and* $\{\hat{\mu}_w^*(v,q), w \in W\}$ *are the unique equilibrium link flows and the generalized equilibrium O-D travel times of the special symmetric UE traffic assignment problem* (12) *for any given* (v,y)*, respectively.*

Although the gap function $G_\alpha(v,q)$ does not possess an explicit expression, Proposition 2 clearly indicates its function value and gradient can be easily evaluated by employing any symmetric UE traffic assignment method.

3. A SOLUTION METHOD

Since the derived single level single level continuously differentiable optimization model (14)-(17) is still a non-convex minimization problem**,** we can employ the augmented Lagrangian algorithm. In view of the non-linearity and implicitly of the gap function shown in the constraint (17), we are able to use a partial penalty function to incorporate such a constraint into the objective function, so that the induced minimization problem possesses network flow conservation equations.

Here, we only present a framework of the augmented Lagrangian algorithm for solving the single level minimization problem (14)-(17) based on the augmented Lagrangian function defined below.

$$\hat{L}(v,q,\lambda,\rho) = F(v,q) + \left(-\lambda G_\alpha(v,q) + \frac{\rho}{2} G_\alpha^2(v,q)\right) \tag{20}$$

where λ is the Lagrangian multiplier with respect to constraint (17) and $\rho > 0$ is the penalty parameter.

The Augmented Lagrangian Algorithm

Step 0. (*Initialization*) Select an initial (v_0, q_0) with $v_0 \in \Omega(q_0)$, the positive constant parameters $\alpha > 1$, $\beta > 1$ and $\gamma \in (0,1)$ and a positive definite diagonal matrix M. Choose an initial multiplier $\lambda_0 \geq 0$, initial penalty parameter $\rho_0 > 0$ and an error tolerance parameter $\varepsilon \geq 0$. Let $k = 0$.

Step 1: (*Inner loop procedure*) Solve the following sub-problem:

$$\min_{v,q} \hat{L}(v,q,\lambda_k,\rho_k) \tag{21}$$

subject to

$$q_w \geq 0, w \in W \tag{22}$$

$$v \in \Omega(q) \tag{23}$$

Let (v_{k+1}, q_{k+1}) denote the optimal solution of the above minimization problem.

Step 2: (*Update parameters*) If $G_\alpha(v_{k+1}, q_{k+1}) \le \varepsilon$, then terminate. Otherwise, perform the operations as follows:

(i) If $G_\alpha(v_{k+1}, q_{k+1}) \le \gamma G_\alpha(v_k, q_k)$, then set

$$\lambda_{k+1} = \lambda_k - \rho_k G_\alpha(v_{k+1}, q_{k+1}) \tag{24}$$

$$\rho_{k+1} = \rho_k \tag{25}$$

(ii) If $G_\alpha(v_{k+1}, q_{k+1}) > \gamma G_\alpha(v_k, q_k)$, let

$$\rho_{k+1} = \beta\rho_k \tag{26}$$

$$\lambda_{k+1} = \lambda_k \tag{27}$$

Let $k := k+1$, and go to Step 1.

For the nonlinear programming problem (21)-(23) in Step 1, according to Proposition 2, the gradient of the objective function can be evaluated by applying a user equilibrium traffic assignment procedure for the problem (7)-(10) with linear travel cost functions defined by eqn. (13).

4. NUMERICAL TEST

So far we have theoretically shown that there exists a single level continuously differentiable minimization model and the augmented Lagrangian algorithm incorporating with an asymmetric UE traffic assignment method is workable for solving the asymmetric UE-based O-D matrix estimation problem. In order to numerically demonstrate the proposed model and algorithms, two O-D matrix estimation examples used by Yang (1995) and Florian and Chen (1995), respectively, are tested.