A Fluid Model for the Anticipatory Route Guidance Problem

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1 Introduction

Advanced traveler information systems attempt to provide to some or perhaps all the travelers with messages to help them make better travel decisions. As a result, traveler information systems are distinguished on the basis of the type of information they provide in messages. In this paper we focus on *anticipatory* systems. These use real-time measurements to forecast travel conditions in the near-term future and present messages based on these predictions. As a result, a traveler can make a decision based on what conditions are likely to be at network locations at the time he/she will actually be there.

Nevertheless, when many travelers receive guidance, their reactions to it may affect traffic conditions significantly. As a result, the key issue in generating guidance messages based on traffic condition forecasts is to ensure that travelers' reactions to the guidance do not invalidate the forecasts the guidance is based on, and render the guidance irrelevant or worse. Anticipatory guidance is *consistent* when the forecasts on which it is based are indeed experienced by travelers after they react to it. Generating consistent guidance is called the Anticipatory Route Guidance (ARG) problem. In this paper we study the ARG problem through a dynamic network loading (DNL) formulation.

Traffic simulation models, such as DYNASMART (Mahmassani *et al.* [7]), MITSIM and DynaMIT (Ben-Akiva *et al.* [1] and Bottom [3]), represent a variety of route guidance technologies. Bovy and van der Zijpp [4] and Bottom [3] have been the first to model the ARG problem.

2 Notation

The physical traffic network is represented by a directed network $G = (N, A)$, where N is the set of nodes and A is the set of directed links. N_1 denotes the set of origin nodes, N_2 the set of control nodes (i.e. where vehicles may receive guidance messages), and P the set of all paths. In the following, p denotes a path between origin node r and destination node s, and K_{rs} denotes the subset of paths between O-D pair (r, s).

Path variables:

$RS(p)$	:	(O-D) pair associated with path p;
p^1	:	first link of path p;
p^l	:	last link of path p;
$f_p(t)$	:	departure flow rate on path p at time t;
$f^{RS(p)}(t)$	:	flow rate of O-D pair $RS(p)$ departing the origin at time t;
$f^{n,RS(p)}(t)$	:	flow rate of O-D pair $RS(p)$ exiting node n at time t;
f	:	vector of path departure flow rate functions $f_p(\cdot)_{p\in P}$ for all times t;
$\beta_{np}(t)$	:	path splitting rate, i.e., the fraction of flow exiting node n via path p at time t;
β	:	vector of path splitting rate functions $\beta_{np}(\cdot)$;
$S_p(t, f)$	:	travel time on path p departing at time t for a vector of flows $f(\cdot)$.

Link variables:

$head(a)$	:	head node of link a; if $a = (m, n)$, $head(a) = m$
$tail(a)$	:	tail node of link a; if $a = (m, n)$, $tail(a) = n$
$u_a(t)$	:	entrance flow rate of link a at time t;
$v_a(t)$	:	exit flow rate of link a at time t;
$U_a(t)$	:	cumulative entrance flow of link a during interval $[0, t]$;
$V_a(t)$	:	cumulative exit flow of link a during interval $[0, t]$;
$X_a(t)$	:	load (number of vehicles) on link a at time t;
$D_a(y)$	:	traversal time function of link a, where y is the number of vehicles on link a;
$s_a(t)$	:	exit time of a flow entering link a at time t
	$=$	$t + D_a(X_a(t))$.

Link-path flow variables:

(a, p)	:	a link-path pair;
δ_{ap}	$=$	1 if link a belongs to path p, and 0 otherwise;
$u_{ap}(t)$	:	entrance flow rate on link a traveling on path p at time t;
$v_{ap}(t)$	:	exit flow rate on link a traveling on path p at time t;
$U_{ap}(t)$	:	cumulative entrance flow on link a traveling on path p during interval $[0, t]$;
$V_{ap}(t)$	:	cumulative exit flow on link a traveling on path p during interval $[0, t]$;
$X_{ap}(t)$	:	partial load on link a at time t induced by flow on path p.

Below, we perform a network transformation that will enable us to define decision variables that are equivalent to the path-node splitting rates $\beta_{np}(t)$. This network transformation is the basis of our approach.

We divide every path $p \in K_{rs}$ that goes through a control node into subpaths in the following manner. Subpaths can either (i) originate at r and end at the first control node on path p, or (ii) originate at a control node and end at the following control node on path p, or (iii) originate at the last control node on path p and end at s. Let P_1 denote the set of subpaths we have just described. Note that this definition of subpaths allows P_1 to contain several copies of the same subpath. However, these copies come from different paths. Let P_2 denote the set of paths that do not go through control nodes (however, a path in P_2 might originate at a control node).

Let $\overline{P} = P_1 \cup P_2$. Also, let $\overline{P}_1$ be the set of subpaths that originate at control nodes and hence receive information about the network conditions. It follows that $\overline{P}_1 = \{p \in \overline{P} | head(p^1) \in N_2\}$, where $head(p^1)$ denotes the head of the first link of path p, i.e. the origin node of path p. Let $\overline{P}_2 = \overline{P} \setminus \overline{P}_1$.

Let $\widehat{p}(p)$ denote the path containing subpath p (this could be path p itself if $p \in P_2$). $s(p)$ will denote the first subpath on path p (again, this could be path p itself if $p \in P_2$). Finally, $f_{p\widehat{p}(p)}(t)$ denotes the subpath flow rate at time t on subpath p of path $\widehat{p}(p)$.

3 Feasibility Conditions of the ARG Problem

The objective of this section is to mathematically define the feasible region of the ARG problem. The region corresponds to the set of feasible network flows and travel times, as established by solutions to dynamic network loading (DNL) problems. In the context of the ARG problem, the DNL problem consists of determining the time-varying link, subpath and path flows and travel times that result from the movement of given O-D flows over the network in accordance with particular path or subpath splitting rates at the origins and at intermediate control nodes. For a fixed network structure and set of O-D flows, the ARG DNL problem can thus be viewed as a map from the domain of path splitting rates to the range of network flows and travel times.

Similar to the DNL maps proposed by Friesz *et al.* [5], by Wu *et al.* [8] and by Kachani [6] in the context of the Dynamic User-Equilibrium (DUE) problem, the ARG DNL map is formulated as a system of equations expressing link dynamics, flow conservation, flow propagation, non-negativity and boundary constraints. Unlike the DUE DNL problem, in which flows departing from their origin on a particular path follow that path without change to their destination, the ARG DNL problem allows flow to switch from one subpath to another at intermediate locations. As a result, the ARG DNL map has two added features: (i) the model is formulated in terms of subpath flow rates instead of path flow rates, and (ii) the path-node splitting rates are explicitly used.

Link dynamics equations: The link dynamics equations express the relationship between the flow variables of a link as follows:

$$\frac{dX_{ap}(t)}{dt} = u_{ap}(t) - v_{ap}(t), \qquad \forall (r,s), \forall p \in K_{rs}, \forall a \in p. \tag{1}$$

Flow conservation equations: For every link a that has a head node that is neither an origin node nor a control node (i.e. $head(a) \in N \setminus (N_1 \cup N_2)$), the flow conservation equations can be expressed as

$$u_{ap}(t) = v_{a'p}(t), \tag{2}$$

where a' is the link preceding link a on path p. For every link a that has a head node n that is an origin node (i.e. $n \in N_1$) and for all paths p originating at n, the flow conservation equations can be expressed as

$$u_{ap}(t) = f_{s(p)p}(t)\beta_{np}(t)f^{RS(p)}(t),$$

where the (O-D) pair departure flow rates $f^{RS(p)}(t)$ are given.

For every link a that has a head node n that is a control node (i.e. $n \in N_2$) and for all paths p that do not originate at n, the flow conservation equations can be expressed as

$$u_{ap}(t) = f_{p\widehat{p}(p)}(t) = \beta_{np}(t)f^{n,RS(p)}(t),$$

where $f^{n,RS(p)}(t) = \sum_{\overline{p}\in RS(p),\overline{a}\in\overline{p}|tail(\overline{a})=n} v_{\overline{ap}}(t)$.

Link-path flow relationships: The following relationships express the fact that at each time t the link flow variables are the sum of their corresponding link-path variables:

$$u_a(t) = \sum_{p|a\in p} u_{ap}(t), \quad v_a(t) = \sum_{p|a\in p} v_{ap}(t),$$

$$U_a(t) = \sum_{p|a\in p} U_{ap}(t), \quad V_a(t) = \sum_{p|a\in p} V_{ap}(t), \tag{3}$$

$$X_a(t) = \sum_{p|a\in p} X_{ap}(t), \quad \forall(r,s), \forall p \in K_{rs}, \forall a \in p.$$

Flow propagation equations: Flow propagation equations are used to describe the flow progression over time. Note that a flow entering link a at time t will exit the link at time $s_a(t)$. Therefore, at time t, the cumulative exit flow of link a should be equal to the integral of all inflow rates which would have entered link a at some earlier time ω and exited link a by time t. This relationship is expressed by the following equation:

$$V_{ap}(t) = \int_{\omega\in W} u_{ap}(\omega)d\omega, \qquad \forall(r,s), \forall p \in K_{rs}, \forall a \in p, \tag{4}$$

where $W = \{\omega : s_a(\omega) \leq t\}$.

Link exit time functions $s_a(t)$ are obtained from link travel time functions through $s_a(t) = t + D_a(X_a(t))$.

Non-negativity constraints: We further assume that the departure path flow rates are non-negative: That is, $f_p(\cdot) \geq 0$, $\forall(r,s), \forall p \in K_{rs}$.

Boundary equations: Since we assume that the network is empty at $t = 0$, the following boundary conditions are required: $U_{ap}(0) = 0$, $V_{ap}(0) = 0$, $X_{ap}(0) = 0$, $\forall(r,s), \forall p \in K_{rs}, \forall a \in p$.

4 A Variational Inequality Formulation

Similarly to Friesz *et al.* [5], who developed variational inequality formulations for the Dynamic User-Equilibrium problem in terms of *path flow rates*, we can formulate the ARG problem in terms of *subpath flow rates*. There are two types of travelers in the network:

• "Uninformed" travelers who travel on subpaths $p \in \overline{P}_2$ and, as a result, do not receive any information. These travelers have some estimate $\widehat{f}$ of the vector of time-dependent flows in the network (these could be, for example, based on observations by these travelers in prior trips). We assume that uninformed travelers utilize these flow estimates to determine the vector of path travel times $S(t, \widehat{f})$. Furthermore, these travelers use the path time estimates to select paths that minimize their O-D travel times. The simplest case occurs when $\widehat{f}$ corresponds to the vector of free-flow path travel times $S(t, 0)$. In this case, travelers follow the static shortest path from their origin to their destination.
• "Informed" travelers who travel on subpaths $p \in \overline{P}_1$ (that originate at control nodes) and as a result receive full information about the network traffic conditions. We assume that these travelers behave "rationally" in the sense that they select paths that minimize their flow-dependent dynamic travel times.

In what follows, we consider these two categories of travelers traveling in the same network. Note that the behavior of the uninformed travelers is unaffected by anything that the informed travelers do, but not conversely. The above two classes of travelers allow us to reformulate the ARG problem as the following variational inequality problem: Find a vector of time-dependent flows $f^\star \in F(ARG)$ satisfying

$$\sum_{r,s} \sum_{p \in \overline{P}_2} \int_0^{T_\infty} S_p(w, \widehat{f})(f_{p\widehat{p}(p)}(w) - f^\star_{p\widehat{p}(p)}(w))dw + \tag{5}$$
$$\sum_{r,s} \sum_{p \in \overline{P}_1} \int_0^{T_\infty} S_p(w, f^\star)(f_{p\widehat{p}(p)}(w) - f^\star_{p\widehat{p}(p)}(w))dw \quad \geq 0, \qquad \forall f \in F(ARG).$$

The first part of this variational inequality formulation describes the uninformed travelers while the second part describes the informed ones.

In this formulation, the problem data consist of the network structure and link travel time (performance) functions $D_a(X_a)$. The input variables are the (O-D) pair departure flow rates $f^{rs}(t)$. The unknown variables that we wish to determine are the link and path entrance flow rates $u_a(\cdot)$ and $f_p(\cdot)$, the link exit flow rates $v_a(\cdot)$, the link cumulative entrance and exit flows $U_a(\cdot)$ and $V_a(\cdot)$, the link loads $X_a(\cdot)$, and the link and path exit time functions $s_a(\cdot)$ and $S_p(\cdot)$.

Theorem 1 ([2])
Assume that the following conditions hold:
(1) The delay functions $D_a(.)$ are continuously differentiable, and there exist two non-negative constants B_{1a}, B_{2a} such that for every link load level X_a, $0 \leq B_{1a} \leq D'_a(X_a) < B_{2a}$.
(2) The link entrance flow rate function $u_a(.)$ is Lebesgue integrable, non-negative and bounded from above by $\frac{1}{B_{2a} - B_{1a}}$.
Then, the ARG Model has an optimal solution.

Bottom, Kachani and Perakis [2] propose a method for solving this variational inequality formulation and report some computational results.

In the special case of the DUE, the ARG problem simplifies significantly. The previous variational inequality becomes: Find a vector of flows $f^\star \in F(DUE)$ satisfying

$$< S(f^\star), f - f^\star > \quad \geq 0, \qquad \forall f \in F(DUE). \tag{6}$$

5 Conclusions

The overall goal of this paper is to establish and analyze a general analytical framework of the ARG problem. This research aims to provide a deeper understanding as to why traffic guidance simulators work in practice. The development and analysis of this framework will allow us to achieve the following:

• Describe and analyze an important generalization of the standard dynamic network loading (DNL) problem, in which origin-destination flows may change paths at intermediate nodes.

• Provide a rigorous analysis of the existence of solutions to the ARG problem under weak assumptions. We believe providing a deeper understanding in this problem will be helpful in improving and understanding why existing simulation software works in practice.

• Compare the ARG problem to the dynamic and static traffic equilibrium problems.

• Propose and analyze practical solution algorithms for solving the problem.

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