# Heterogeneous Users and Variable Road Pricing: Model and Algorithm for the Bi-Criterion Dynamic User Equilibrium Problem on Large Networks

Chung-Cheng Lu (cclu@umd.edu)

Hani S. Mahmassani (masmah@umd.edu, corresponding author)

Department of Civil & Environmental Engineering
University of Maryland
3128 Jeong H. Kim Engineering Building
College Park, Maryland 20742

## 1. Introduction

To support the planning, operation, and evaluation of various dynamic road pricing schemes, particularly, on large-scale networks, dynamic user equilibrium (DUE) network assignment models are often applied to predict path choices and the resulting network flow patterns, which in turn form the basis for assessing the economic and financial impacts or benefits of proposed toll facilities or schemes. DUE models for dynamic road pricing applications should essentially be able to (i) capture traffic flow dynamics and spatial and temporal vehicular interactions, (ii) adhere to the time-dependent generalization of Wardrop's first principle (Wardrop, 1952), i.e. so-called DUE conditions, (iii) provide the basis for an algorithm that exhibits better performance (solution quality and computational effort) than commonly used algorithmic schemes on practical networks, (iv) address the heterogeneous user preference of path choice in response to toll charges, and (v) be deployable on large-scale networks. This paper describes a bi-criterion dynamic user equilibrium (BDUE) model and solution algorithm that meet the above requirements.

## 2. Assumptions, Definition, and Problem Statement

Given a time-dependent network G = ($N$, $A$), where $N$ is the set of nodes and A is the set of directed links ($i$, $j$), $i \in N$ and $j \in N$. The time period of interest (planning horizon) is discretized into a set of small time intervals, $S = \{t_0, t_0 + \sigma, t_0 + 2\sigma, \ldots, t_0 + M\sigma\}$, where $t_0$ is the earliest possible departure time from any origin node, $\sigma$ is a small time interval during which no perceptible changes in traffic conditions and/or travel cost occur, and $M$ is a large number such that the intervals from $t_0$ to $t_0+M\sigma$ cover the planning horizon $S$. Without loss of generality, associated with each arc ($i$, $j$) and departure time interval t are two essential time-dependent arc travel impedances: time ($d_{ij}(t)$) and cost ($c_{ij}(t)$), which are required to travel from node $i$ to node $j$ when departing at time $t$ from node $i$, and would be minimized simultaneously in trip-makers' path choice decision framework. Note that $d_{ij}(t)$ may include both non-congested travel time and delay, while some other cost-related arc attributes can be considered in $c_{ij}(t)$.

By assuming that path travel disutilities are additive of their respective link travel disutilities, the experienced path generalized cost perceived by a trip-maker with VOT $\alpha$ is defined as path travel cost plus path travel time weighted by the $\alpha$. The VOT $\alpha$ relative to each trip represents how much money the trip-maker is willing to trade for a unit time saving. To reflect heterogeneity of the population, the VOT in this study is treated as a continuous random variable distributed across the population of trip-makers, with the density function $\phi(\alpha)>0$, $\forall\alpha\in[\alpha^{\min},\alpha^{\max}]$ and $\int_{\alpha^{\min}}^{\alpha^{\max}}\phi(\alpha)d\alpha=1$, where the feasible range of VOT is defined by the closed interval $[\alpha^{\min},\alpha^{\max}]$. Note that the distribution of VOT is assumed given, and can be estimated from survey data. The time-dependent origin-destination (OD) demand for the entire feasible range of VOT over the planning horizon (i.e. number of trips for each OD pair, each departure time interval and each possible value of VOT) is also known a priori.

The key behavioral assumption for the path choice decision is as follows: in a disutility-minimization framework, each trip-maker chooses a path that minimizes the generalized cost (i.e. disutility). Accordingly, the bi-criterion dynamic user equilibrium (BDUE) that extends Wardrop's first (or UE) principle (Wardrop, 1952), is stated as: *For each OD pair and for each departure time interval, every trip-maker cannot decrease the experienced generalized trip cost with respect to that trip's particular VOT $\alpha$ by unilaterally changing paths.* This definition can be also viewed as the dynamic extension of Dial's bi-criterion equilibrium traffic assignment (Dial, 1996) or Leurent's cost versus time equilibrium (Leurent, 1993). Given the assumptions and definition above, this study aims at solving the BDUE traffic assignment problem, under a given time-dependent road pricing scheme, to obtain the time-dependent path flow pattern satisfying the BDUE condition. Specifically, the focus is to determine the BDUE path flows (routing policies) in a vehicular network for each OD pair, each departure time interval and all possible values of time.

## 3. Model Formulation

Notations and Variables

$r_{od}^{\tau}(\alpha)$ number of trips with VOT $\alpha$ departing from *o* to *d* in departure time interval $\tau$.

$r_{od}^{\tau}$ number of trips departing from *o* to d in time interval $\tau$, and $r_{od}^{\tau}=\int_{\alpha^{\min}}^{\alpha^{\max}}r_{od}^{\tau}(\alpha)d\alpha$.

$r_{odp}^{\tau}(\alpha)$ number of trips with VOT $\alpha$ departing from *o* to *d* in time interval $\tau$ that are assigned to path $p\in P(o, d, \tau)$.

$r(\alpha)$ the class-specific time-varying path flow vector for the trips with VOT $\alpha$; i.e. $r(\alpha)\equiv\{r_{odp}^{\tau}(\alpha), \forall o, d, \tau, \text{ and } p\in P(o, d, \tau)\}$.

$r$ the time-varying path flow vector for the trips with all possible values of time; i.e. $r\equiv\{r(\alpha), \forall\alpha\in[\alpha^{\min}, \alpha^{\max}]\}$.

***TT*** the vector of experienced path travel times.

***TC*** the vector of experienced path travel costs.

Let $\Omega(\alpha)\equiv\{r(\alpha)\}$ be the feasible set of path flow vectors $r(\alpha)$ satisfying the path flow conservation and non-negativity constraints:

$$\sum_{p\in P(o,d,\tau)}r_{odp}^{\tau}(\alpha)=r_{od}^{\tau}(\alpha),\ \forall o, d, \text{ and } \tau, \tag{1}$$

$$r_{odp}^{\tau}(\alpha)\geq 0,\ \forall o, d, \tau, \text{ and } p\in P(o, d, \tau). \tag{2}$$

Solving for the BDUE flow pattern $r^*$ is equivalent to finding the solution of a system of variational inequalities $r^*(\alpha)\in\Omega(\alpha)$ such that

$$G(\alpha, r^*)^T \circ (r^*(\alpha) - r(\alpha)) \le 0,\ \forall\ r(\alpha)\in\Omega(\alpha),\ \forall\alpha\in[\alpha^{\min}, \alpha^{\max}], \tag{3}$$

where $\circ$ is the inner product in $I^m$, and $G(\alpha, r^*)$ is the path generalized cost vector perceived by the trips with VOT $\alpha$ and evaluated at flow pattern $r^*$. Since (3) is only required to hold on $[\alpha^{\min}, \alpha^{\max}]$, it can be further represented by the following (possibly) infinite dimensional VI (see e.g. Marcotte and Zhu, 1997): find $r^* \equiv \{r^*(\alpha),\ \forall\alpha\in[\alpha^{\min}, \alpha^{\max}]\}$ and $r^*\in\Omega$ such that

$$G(r^*)^T \circ (r^* - r) \le 0,\ \forall\ r\in\Omega \tag{4}$$

where $G(r^*) \equiv\{G(\alpha, r^*),\ \forall\alpha\in[\alpha^{\min}, \alpha^{\max}]\}$, and $\Omega=\{r\}=\{\Omega(\alpha), \forall\alpha\in[\alpha^{\min}, \alpha^{\max}]\}$.

## 4. Solution Algorithm

Since the infinite dimensional VI formulation of the BDUE problem uses path-related variables, a set of feasible paths on which the time-varying OD demands is to be equilibrated is required. It is generally very difficult, if not impossible, to enumerate the complete set of feasible paths of all OD pairs for a road network of practical size. To avoid explicit enumeration of all possible paths, this study uses a column generation-based approach to generate a representative subset of paths that have competitive travel times. The column generation-based approach augments, in the outer loop, the subset of the extreme efficient paths and solves, in the inner loop, the restricted multi-class dynamic user equilibrium (RMDUE) sub-problem defined by the current subset of extreme efficient paths. Embedded in this algorithmic framework is a simulation-based dynamic traffic model that captures traffic dynamics and determines experienced path generalized costs $G(r)$ for any given path flow pattern $r$; that is traffic flow propagations and the vehicular spatial and temporal interactions are addressed through the traffic simulation instead of analytical calculations. By and large, the original BDUE problem, finding a DUE path flow pattern resulting from the interactions of (possibly) infinite number of user classes, is solved in this algorithmic framework as series of approximate RMDUE sub-problems to progressively find BDUE solutions.

**Initialization**

0. Input: (1) a time-dependent OD demand matrix for the entire feasible range of VOT over the planning horizon ($r_{od}^{\tau}(\alpha)$, $\forall o, d, \tau$, and $\alpha\in[\alpha^{\min},\alpha^{\max}]$), (2) a set of time-dependent link tolls, (3) VOT density and distribution functions ($\phi(\alpha)$ and $\Phi(\alpha)$, $\forall\alpha\in[\alpha^{\min},\alpha^{\max}]$), and (4) initial paths and path assignment.
1. Set the iteration counter of outer loop $k = 0$. Perform a multi-class dynamic network loading (MDNL) with the initial path assignment. Obtain time-dependent link travel times and experienced path travel times and costs (i.e. ***TT*** and ***TC***) from the traffic simulator.

**Outer Loop – generating extreme efficient path set**

2. Solve the bi-criterion time-dependent least generalized cost path (tree) problem to obtain the set of extreme efficient paths, their corresponding generalized costs ($\pi^k$), and breakpoints that partition the entire VOT interval and define the multi-user classes.
3. Convergence checking: if (a) there is no new path found or (b) $k = K_{max}$ (maximum number of outer iterations) then stop; otherwise start the inner loop (go to step 4).

**Inner Loop – solving the RMDUE sub-problem**

4. Set the iteration counter of inner loop $l$ = 0; read the output of step 2: $\pi^l$ and VOT breakpoints, as well as the current path set (and ***TT*** and ***TC***) and path assignment ($r^l$).
5. Update path assignment: determine path assignment $r^{l+1}$ by using the descent direction method. Set $l = l + 1$.
6. MDNL: perform a MDNL with the new path assignment $r^l$. Obtain experienced path travel times and costs (i.e. ***TT*** and ***TC***) from the traffic simulator.
7. Find, in the existing path set, the least generalized cost path for each $(o,d,\tau)$ and each user class $u$, defined based on the breakpoints obtained in step 2.
8. Convergence checking: if $|Gap(r^l)–Gap(r^{l-1})| \leq \varepsilon$ (a preset convergent threshold; $Gap(r^k)$ will be defined in the following subsection) or $l = L_{max}$ (maximum number of inner iterations) then return to the step 2 (the outer loop) with current link travel times, path generalized costs $G(r^l)$ and $r^l$, and set $k = k+1$; otherwise go back to step 5.

### 4.1 Outer Loop – Generating Extreme Efficient Path Set

This study adopts the bi-criterion time-dependent least generalized cost path (BTDLGCP) algorithm developed by Mahmassani et al. (2005) to find the set of time-dependent extreme efficient paths and the corresponding set of breakpoints (i.e. values of time $\alpha = \{\alpha^0, \alpha^1, ..., \alpha^I \mid \alpha^{\min} = \alpha^0 < \alpha^1 < ... < \alpha^i < ... < \alpha^I = \alpha^{\max}\}$ that partitions the entire feasible range of VOT and hence defines the multiple user classes of trips, where each class $u^i$ covers the trips with VOT $\alpha \in [\alpha^{i-1}, \alpha^i)$, $i = 1,\ldots I$. Starting from the lowest possible VOT, the BTDLGCP algorithm continuously solves for the time-dependent least generalized cost path (TDLGCP) tree rooted at each destination for a given VOT subinterval and determines the upper bound of that VOT subinterval, for which the TDLGCP tree remains optimal, until reaching the highest possible of VOT. If there is no new path found for each $(o,d,\tau)$ and each user class $u$, or the outer loop iteration counter $k$ equals $K_{max}$ (maximum number of outer iterations) then terminate the algorithm; otherwise start the inner loop with the output of BTDLGCP algorithm: $\pi^k$ and $\alpha$, as well as current path set (and ***TT*** and ***TC***) and path assignment $r^k$ .

### 4.2 Inner Loop – Solving the RMDUE Sub-problem

<u>4.2.1 The RMDUE Sub-problem</u>

With the set of breakpoints $\alpha$ that determines the (finite number of) multiple user classes ($u^i$, $i = 1,\ldots I$), the sub-problem defined by a (currently available) subset of the feasible time-dependent extreme efficient paths and solved in the inner loop of the column generation-based algorithmic framework can be considered as a restricted multi-class dynamic user equilibrium (RMDUE) problem by following the terminology often adopted in the literature. Solving the RMDUE problem aims at finding a finite-dimensional multi-class path flow vector so that for each $(u^i, o, d, \tau)$ every trip-maker cannot decrease the experienced generalized trip cost by unilaterally changing paths.

<u>4.2.2 A Descent Direction Method for Solving the RMDUE sub-problem</u>

This study adapts the descent direction method proposed by Lu and Mahmassani (2006) to solve the RMDUE sub-problem and update path assignments in the inner loop of the column generation-based BDUE solution algorithm. The descent direction method is a projection type algorithm that decomposes the RMDUE problem into many $(u^i, o, d, \tau)$ sub-problems and solves each of them by adjusting time-varying OD flows between (all) non-least generalized cost paths and the least generalized cost path(s). Given a feasible solution $r^l$ in an inner loop iteration $l$, the method features the following form:

$$r^{l+1} = P_{\Omega}[r^l - \rho^l \times Dir^l] = P_{\Omega}[r^l - \rho^l \times \frac{r^l \times (G(r^l) - \pi(r^l))}{G(r^l)}], \quad (5)$$

where $\rho^l \in (0,1)$ is the step size in iteration $l$, $-Dir^l$ is the descent direction, and $\pi(r^l)$ is the vector of least generalized path costs evaluated at $r^l$. $P_{\Omega}[u]$ denotes the unique projection of vector $u$ onto $\Omega$ (the set of feasible multi-class path flow vectors $r$). Based on Eq.(10), the new path assignment $r^{l+1}$ is obtained by updating the current path assignment $r^l$ along the descent direction ($-Dir^l$) with a move size $\rho^l$. This path assignment updating scheme is intuitively based on the fact that travelers farther from the equilibrium and on paths with larger flow rates are more strongly inclined to change path than those on paths with smaller flow rates and with travel cost closer to the minimal cost.

#### 4.2.3 Multi-Class Dynamic Network Loading Using Traffic Simulator

By the BDUE definition, all trips in a network are equilibrated in terms of actual experienced path generalized costs, consisting of experienced path times and path costs, so it is necessary to determine the experienced path generalized costs $G(r)$ for a given multi-class path flow vector $r$. To this end, the simulation-based dynamic traffic (network loading) model – DYNASMART (Jayakrishnan et al., 1994) is employed to evaluate a path assignment $r$ and to obtain $G(r)$ and time-dependent link travel times used in the path generation step. It should be noted that the algorithm is independent of the specific dynamic traffic model selected; any particle-based (microscopic or mesoscopic) dynamic traffic model capable of capturing complex traffic flow dynamics can be embedded into the proposed algorithm.

#### 4.2.4 Convergence Checking Using Gap Values

This study extends the gap-based criterion proposed by Lu and Mahmassani (2006) to the BDUE context and defines the multi-class version of the gap function as the following:

$$Gap(r^l) = \sum_{u^i}\sum_{o}\sum_{d}\sum_{\tau}\sum_{p\in P(u^i,o,d,\tau)} r^{\tau,l}_{odp}(u^i) \times [G^{\tau}_{odp}(u^i, r^l) - \pi^{\tau}_{od}(u^i, r^l)] \quad (6)$$

Note that, $Gap(r^l)$ provides a measure of the violation of the BDUE conditions in terms of the difference between the total actual experienced path generalized cost and the total least generalized cost evaluated at any given multi-class path flow pattern $r$. The difference vanishes when the path flow vector $r^*$ satisfies the BDUE conditions.

## 5. Preliminary Numerical Results

The proposed BDUE algorithm is implemented and tested on the Irvine (California, USA) network, consisting of 326 nodes (70 of them are signalized), 626 links, and 61 traffic analysis zones (Mahmassani et al. 2003). To create hypothetic dynamic road pricing scenarios, one lane of a portion (about 1 mile) of the I-405 westbound freeway is converted to the toll road, along with an additional new toll lane. The two toll lanes have the same length as the (remaining) three regular lanes but a 10-mile higher posted speed limit (and hence higher capacity) than the regular lanes. Table 1 lists the three simple dynamic pricing scenarios tested in the experiments conducted on the Irvine network. These three pricing scenarios have the same four pricing periods but different toll levels representing low, middle, and high toll scenarios, respectively. The experimental results are presented in Table 2. These small gap values indicate that the BDUE algorithm is able to find close-to-BDUE solutions.

Table 1 Dynamic road pricing scenarios tested on the Irvine network

| Pricing Scenario | Period 1 (7:00-7:30AM) | Period 2 (7:30-8:00AM) | Period 3 (8:00-8:30AM) | Period 4 (8:30-9:00AM) |
|---|---|---|---|---|
| 1 | \$0.10 | \$0.20 | \$0.30 | \$0.15 |
| 2 | \$0.20 | \$0.30 | \$0.40 | \$0.25 |
| 3 | \$0.30 | \$0.40 | \$0.50 | \$0.35 |

Table 2 Experimental results on the Irvine network

| | Gap($r$) | | |
|---|---|---|---|
| Iteration | Scenario 1 | Scenario 2 | Scenario 3 |
| 1 | 1669.0 | 1846.6 | 2224.7 |
| 2 | 1478.6 | 1851.8 | 1444.9 |
| 3 | 1260.1 | 737.3 | 873.0 |
| 4 | 559.1 | 551.4 | 764.1 |
| 5 | 536.4 | 1034.5 | 483.3 |
| 6 | 485.9 | 657.2 | 614.2 |
| 7 | 917.0 | 703.8 | 395.9 |
| 8 | 452.9 | 458.8 | 695.4 |
| 9 | 724.5 | 809.5 | 451.4 |
| 10 | 310.9 | 287.5 | 333.3 |
| 11 | 698.1 | 785.5 | 452.7 |
| 12 | 320.6 | 863.7 | 295.5 |
| 13 | 312.5 | 316.2 | 314.2 |
| 14 | 291.4 | 300.2 | 290.0 |
| 15 | 602.1 | 285.3 | |
| 16 | 406.1 | 263.2 | |
| 17 | 361.7 | 299.4 | |
| 18 | 262.7 | | |

## References


Dial, R. B. (1996). Bicriterion traffic assignment: basic theory and elementary algorithms. *Transportation Science*, Vol. 30 (2), pp. 93-110.

Jayakrishnan, R., Mahmassani, H. S., and Hu, T.-Y. (1994). An evaluation tool for advanced traffic information and management systems in urban network. *Transportation Research Part C*, Vol. 2 (3), pp. 129-147.

Leurent, F. (1993). Cost versus time equilibrium over a network. *European Journal of Operational Research*, Vol. 71 (2), pp. 205-221.

Lu, C.-C., Mahmassani, H. S. and Zhou, X. (2006). Equivalent gap function-based reformulation and solution algorithm for the dynamic user equilibrium problem. Submitted to *Transportation Research Part B*.

Mahmassani, H. S., Qin, X., Zhou, X., Mahfoud, R. H., Lu, C.-C. and Erdogan, S. (2003). TMC evaluation report for a real-time DTA system prototype: phase 1.5B. Technical Report, Maryland Transportation Initiative, University of Maryland, College Park, USA.

Mahmassani, H. S., Zhou, X., and Lu, C.-C. (2005). Toll pricing and heterogeneous users: approximation algorithms for finding bi-criterion time-dependent efficient paths in large-scale traffic networks. *Transportation Research Record* 1923, pp.28-36.

Marcotte, P. and Zhu, D. L. (1997). Equilibria with infinitely many differential classes of customers. In Complementarity and Variational Problems. State of the Art, *Proceedings of the 13th International Conference on Complementarity Problems*, Jong-Shi Pang and Michael Ferris, eds., SIAM, Philadelphia, pp. 234-258.