Stochastic Network Equilibrium Model under Uncertain Demand

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INTRODUCTION

Evaluating uncertainty of traffic networks is very important for network design. One of the methodology (or method) of assessing the uncertainty theoretically is an equilibrium model that can estimate probability distributions of travel times or traffic flows.

Stochastic User Equilibrium (SUE), introduced by Daganzo & Sheffi (1977), is one of the most important network equilibriums. SUE is regarded as Wardrop's equilibrium (Wardrop, 1952) with route choice based on random utility models. The (route) utility in route choice of SUE has an error term. The interpretation of the error term is disputable. Variance of the error term is constant despite the route's length, and the term does not seem to reflect variation or uncertainty of travel time on the route. The error term should be interpreted as "perceptual" error or effect of the components that are not considered in the model. Furthermore, network flows in SUE is not stochastic but deterministic. SUE cannot treat uncertainty or variation of network flows.

There have been several studies about uncertainty of network flows. Mirchandani & Soroush (1987) assumed that free-flow travel time is random, and proposed a network equilibrium model with probabilistic travel times. Arnott et al. (1991) and Chen et al. (2002) introduced random capacity to network equilibrium. These three studies assumed exogenous randomness. Cascetta (1989) and Cascetta & Canterella (1991) formulated day-to-day dynamics of network flows as a Markov process. The convergent distribution of network flow could be interpreted as network equilibrium with stochastic flow. Watling (2002) extended SUE and presented a second order stochastic network equilibrium. He assumed route choice based on random utility theory and stochastic flow variables. The travel demands are assigned based on the mean cost. Cascetta (1989), Cascetta & Canterella (1991), and Watling (2002) consider stochastic route choice, but the travel demand is fixed. One of the main causes of network uncertainty is variation of travel demands. Stochastic demand should be incorporated into network equilibrium models.

In this study, we assume stochastic demand as well as stochastic route choice, and formulate a stochastic network equilibrium model under stochastic demands. This model enables us to examine network reliability under uncertain demands.

CONDITIONAL FLOW DISTRIBUTION

We assume that a driver chooses a route stochastically. This represents a combination of choices with probabilities. For example, Choice 1 is adopted with probability 0.5 and Choice 2 with 0.5. This type of choice is called mixed strategy in game theory. In Watling (2002), the route choice probabilities are given by the random utility models. In this study, for simplicity, the route choice probabilities are given by the logit model.

Let i (i = 1, 2,…, I) denote an origin-destination (OD) pair in the network and n^i the demand of the ith OD pair. Let j denote a route, where the total number of routes is J and the number of routes linking the ith OD pair is J_i. Let a denote a link, where the total number of links in the network is A.

Assume that each driver who travels between the ith OD pair chooses the jth route with probability p_{ij}; that is, p_{ij} is the probability of choosing the jth route between the ith OD pair. Clearly, $\Sigma_{i=1}^{J_i} p_{ij} = 1$. The joint probability of route flows between the ith OD pair follows a multinomial distribution if the route choice probability is common among drivers (Sheffi, 1985, p. 281, Watling, 2002). That is, $\mathbf{Y}_i \sim \mathrm{Mn}(n_i, \mathbf{p}^i)$ where $\mathbf{Y}_i$ is the vector of random variables of route flows between the ith OD pair when the demand is n_i and which follows a multinomial distribution, $\mathbf{p}_i$ is the vector of route choice probabilities between the ith OD pair, and Mn(·) is a multinomial distribution.

We can obtain the probability of a single route flow as the marginal probability of the multinomial distribution. The probability mass function of the flow on the jth route between the ith OD pair is expressed as:

$$\begin{aligned} f_{\mathrm{Y}_{ij}|n_i}(y_{ij}) &= \sum_{y_{i1}} \cdots \sum_{y_{ij-1}} \sum_{y_{ij+1}} \cdots \sum_{y_{iJ_i}} \frac{n_i!}{\prod_{j=1}^{J_i} y_{ij}!} \prod_{j=1}^{J^i} (p_{ij})^{y_{ij}} \\ &= \frac{n_i!}{y_{ij}!(n_i - y_{ij})!} p_{ij}^{\,y_{ij}} (1-p_{ij})^{n_i - y_{ij}} \end{aligned} \tag{1}$$

where $f_{\mathrm{Y}_{ij}|n_i}(y_{ij})$ is the probability mass function of flow on the jth route between the ith OD pair when the demand is n_i, $n_i!\prod_j p_{ij}^{\,y_{ij}} \big/ \prod_j y_{ij}!$ is a probability mass function of multinomial distribution, y_{ij} is the realized value of the jth route flow, and Y_{ij} is the component of $\mathbf{Y}_i$. This is a binomial distribution. The flow on a single route follows the binomial distribution $\mathrm{Bn}(n_i, p_{ij})$, where Bn(·) denotes a binomial distribution.

STOCHASIC DEMAND

Travel demand variation is one of the main causes of network uncertainty. Stochastic demand should be incorporated. Assume that travel demand follows a negative-binomial distribution, $\mathrm{NgBn}(\alpha, \beta)$. Negative binomial distributions are discrete and always take positive values unlike normal distributions. The probability mass function of the demand, $g_{N_i}(n_i)$, is:

$$g_{N_i}(n_i) = \frac{\Gamma(\alpha_i + n_i)}{\Gamma(\alpha_i) n_i!} \left(\frac{1}{1+\beta_i} \right)^{\alpha_i} \left(\frac{\beta_i}{1+\beta_i} \right)^{n_i} \tag{2}$$

where $\Gamma(\cdot)$ denotes a gamma function, and α_i and β_i are constant parameters and specify the demand of the ith OD pair. The mean and variance of $g_{Ni}(n_i)$ are $\alpha_i \beta_i$ and $\alpha_i \beta_i (1 + \beta_i)$, respectively.

The route flows between the same OD pair are given as a compound distribution of a multinomial distribution and a negative binomial distribution. That is, the route flows in which the demand follows $g_{Ni}(n_i)$ are given by $f_{\mathbf{Y}_i|n_i}(y_i)\, g_{N_i}(n_i)$.

$$f_{\mathbf{Y}i}(\mathbf{y}_i) = f_{\mathbf{Y}_i|n_i}(\mathbf{y}_i)\; g_{N_i}(n_i)$$

$$= \frac{\Gamma(\alpha_i + n_i)}{\Gamma(\alpha_i) n_i!} \left(\frac{1}{1+\beta_i} \right)^{\alpha_i} \left(\frac{\beta_i}{1+\beta_i} \right)^{n_i} \frac{n_i!}{\prod_{j=1}^{J_i} y_{ij}!} \prod_{j=1}^{J_i} p_{ij}^{\;y_{ij}} \tag{3}$$

$$= \frac{\Gamma(\alpha_i + n_i)}{\Gamma(\alpha_i) \prod_j y_{ij}!}\, \xi_{i0}^{\;\alpha_i} \prod_j \xi_{ij}^{\;y_{ij}}$$

where $n_i = \Sigma_j\, y_{ij}$, $\xi_{i0} = 1/(1 + \beta_i)$, $\xi_{ij} = \beta_i\, p_{ij} /(1 + \beta_i)$. This is a negative-multinomial distribution. Means and variances and covariance are given by:

$$\mu_{ij} = \alpha_i \beta_i p_{ij}$$
$$\sigma_{ij}^{\;2} = \alpha_i \beta_i p_{ij} (1 + \beta_i p_{ij})$$
$$\sigma_{ij,j'} = \alpha_i \beta_i^2\, p_{ij} p_{ij'}\ .$$

Thus, route flows of each OD pair follow a negative-multinomial distribution under negative-binomial distributed demands.

Each route flow follows a negative binomial distribution. The flow on the jth route between the ith OD pair is given by $\Sigma_{y_{ij}} \cdots \Sigma_{y_{ij-1}}\, \Sigma_{y_{ij+1}} \cdots \Sigma_{y_{iJi}} f_{\mathbf{Y}i}(\mathbf{y}_i)$. The p.d.f., $f_{Y_{ij}}(y_{ij})$, is:

$$f_{Y_{ij}}(y_{ij}) = \frac{\Gamma(\alpha_i + y_{ij})}{\Gamma(\alpha_i)\, y_{ij}!} \frac{(\beta_i\, p_{ij})^{y_{ij}}}{(1 + \beta_i\, p_{ij})^{\alpha_i + y_{ij}}} \tag{4}$$

In order to calculate mean travel time, $p_{a,i}$ is defined as $\Sigma_j\, \delta_{a,ij}\, p_{ij}$. This $p_{a,i}$ means the probability that drivers between the ith OD pair travel on the ath link. Let $X_{a,i}$ denote $\Sigma_j\, \delta_{a,ij}\, Y_{ij}$ and $x_{a,i}$ denote the realized value of $X_{a,i}$. $X_{a,i}$ follows the negative binomial distribution NgBn(α_i, β_i, $p_{a,i}$).

$$f_{X_{a,i}}(x_{a,i}) = \frac{\Gamma(\alpha_i + x_{a,i})}{\Gamma(\alpha_i)\, x_{a,i}!} \frac{(\beta_i\, p_{a,i})^{x_{a,i}}}{(1 + \beta_i\, p_{a,i})^{\alpha_i + x_{a,i}}} \tag{5}$$

MEAN TRAVEL TIME

Let x_a denote the traffic volume on the ath link and X_a its random variable. X_a is $\Sigma_i\, \Sigma_j\, \delta_{a,ij}\, Y_{ij}$, where $\delta_{a,ij}$ takes a value of 1 if the ath link is part of the jth route; otherwise its value is 0. As mentioned in the previous section, the link flow follows a probability distribution. The mean travel time on a link is calculated as follows:

$$E[T_a] = E[t_a(X_a)] = \sum_{x_a=0}^{\infty} t_a(x_a) \cdot f_{X_a}(x_a) \tag{6}$$

where $f_{X_a}(\cdot)$ is the probability mass function of flow on the ath link, T_a is the random variable of travel time on the ath link, $t_a(x_a)$ is the travel time function of the ath link, and E[·] is the expectation operator.

In this study, we adopt a BPR-type travel time function for calculating travel time; $t = t_f(1 + c'(x/C)^b)$ where t is the link travel time, t_f is the free-flow travel time, x is the link flow, and b and c' are positive parameters. For simplicity, we express link travel times as $h + c \cdot x^b$, where b, c, and h are positive constant parameters. When b is an integer (4.0 is usually used), the mean link travel time can be calculated using moment generating functions.

A moment generating function, $M(s)$, is defined as $E[e^{sX}]$ (e.g., Ang & Tang, 1977; Papoulis, 1965). The moment generating function of the sum of independent random variables is the product of their moment generating functions. The ath link flow, X_a, is $\sum_j X_{a,i}$. $X_{a,i}$ is mutually independent because the demand is independent among OD pairs. Let $M_{a,i}(s)$ denote the m.g.f. (moment generating function) of $X_{a,i}$ and $M_a(s)$ denote the m.g.f. of the ath link. $M_a(s) = \Pi_i M_{a,i}(s)$. As a property of the moment generating function, $E[X^b] = d^bM(s)/ds^b\,|_{s=0}$. The mean travel time on the ath link is given as:

$$E[T_a] = h + c \cdot \left. \frac{d^b M_a(s)}{ds^b} \right|_{s=0} \tag{7}$$

where $M_a(s)$ is the moment generating function of the ath link.

The variance of link travel time, $\mathrm{Var}[T_a]$, is $E[T_a^2] - \{E[T_a]\}^2$, and $E[T_a^2]$ is also calculated using moment generating functions.

FORMULATION

Assume that each driver choose a route stochastically based on the logit model as follows:

$$p_{ij} = \frac{\exp\left(-\theta \bar{c}_{ij}(\mathbf{p})\right)}{\sum_{j' \in J_i} \exp\left(-\theta \bar{c}_{ij'}(\mathbf{p})\right)} \quad \forall i \, \forall j \tag{8}$$

where v_{ij} is the mean travel time of the jth route between the ith OD pair, $\mathbf{p}$ the vector of the routes, θ a positive parameter. We can incorporate toll fee, risk attitude and so on into v_{ij}.

Define $\mathbf{g} = (g_{11}, .., g_{1J_1}, g_{21}, \ldots, g_{IJ_I})^{\mathrm{T}}$. The component of $\mathbf{g}$, g_{ij}, is:

$$g_{ij}(\mathbf{p}) = \frac{\exp\left(-\theta \bar{c}_{ij}(\mathbf{p})\right)}{\sum_{j' \in J_i} \exp\left(-\theta \bar{c}_{ij'}(\mathbf{p})\right)} \tag{9}$$

where $\mathbf{p}$ is the vector of all route flows and is $(p_{11}, .., p_{1J_1}, p_{21}, \ldots, p_{IJ_I})^{\mathrm{T}}$.

A logit-based stochastic network equilibrium model can be formulated as a fixed point problem as follows:

$$\mathbf{p} = \mathbf{g}(\mathbf{p}) \tag{10}$$

The above can also be formulated as the following complementary problem:

$$\text{Find } \mathbf{z}^* = \begin{bmatrix} \mathbf{p} \\ \boldsymbol{\kappa} \end{bmatrix} \in R_+^J \times R_+^I$$
$$\text{such that } \langle \mathbf{z}, \mathbf{f}(\mathbf{z}) \rangle = 0,\ \mathbf{f}(\mathbf{z}) \geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0} \tag{11}$$

$$\mathbf{f}(\mathbf{z}) = \begin{bmatrix} \mathbf{0} & -\boldsymbol{\Lambda}^T \\ \boldsymbol{\Lambda} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \boldsymbol{\kappa} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{c}} + \mathbf{ln}(\mathbf{p})/\theta \\ -\mathbf{I} \end{bmatrix} \tag{12}$$

where $\mathbf{p} = (\mathbf{p}_1,..,\mathbf{p}_i,..,\mathbf{p}_I)^{\mathrm{T}}$, $\mathbf{p}_i = (p_{i1},\ldots, p_{iJ_i})^{\mathrm{T}}$, $\mathbf{ln}(\mathbf{p}) = (ln\ p_{11},\ldots,\ ln\ p_{IJ_I})^{\mathrm{T}}$, $\bar{\mathbf{c}}(\mathbf{p})$ $(= (\bar{c}_{11}(\mathbf{p}),\ldots, \bar{c}_{IJ_I}(\mathbf{p}))^{\mathrm{T}})$ mean route travel times and $\boldsymbol{\kappa}$ the vector of minimum mean route travel times, $\boldsymbol{\Lambda}$ OD-route incident matrix, $\langle \mathbf{x}, \mathbf{y} \rangle$ the inner product, $\mathbf{I}$ the unit vector, $\mathbf{0}$ the null vector, T transition for vectors or matrices.

EXAMPLE

As an illustration of the above equilibrium model under stochastic demand, a simple network example is presented, consisting of two OD pairs and three links. Fig. 1 shows the example network. Table 1 presents the distributions of OD demands and Fig. 2 illustrates the probability mass functions. The link travel time functions are given in Table 2.

In order to consider drivers' risk attitude, route disutility, U_j, in the logit model is defined as $E[T_{ij}] + \eta\sqrt{\mathrm{Var}[T_{ij}]}$ instead of $E[T_{ij}] = \bar{c}_{ij}$. We set the diversion parameter, θ, and the risk attitude parameter, η, at both 1.0.

We could obtain the route choice probabilities by solving the problem written in the previous section. Table 3 shows the results of the example network. Link 1 and Link 2 are the same link. The (mean) flow on Link 1 is greater than that on Link 2, and the S.D. and variance on Link 1 are greater than those on Link 2. The capacity of Link 3 is half of Link 1 and Link 2, and the flow on Link 3 fluctuates more largely than Link 1 and Link 2. So, S.D.

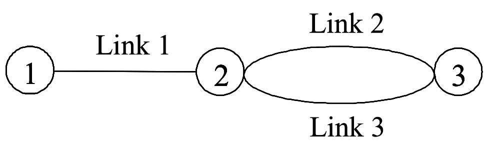

Fig. 1. Example Network

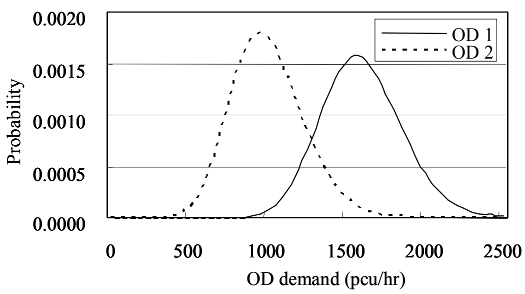

Fig. 2. Demand's Probability Function

Table 1. Demand

	OD 1 between node 1&3	OD 2 between node 2&3
distribution	NgBn(40, 40)	NgBn(20, 50)
mean	1600	1000
variance	65600	51000
S.D.	256.1	225.8

Table 2. Travel Time Function

	Free-flow travel time	Capacity
Link 1	10	2000
Link 2	10	2000
Link 3	5	1000

Table 3. The Results on Link Travel Times

	Link 1	Link 2	Link 3
Mean	16.56	14.40	13.40
S.D.	2.11	1.18	2.24
Variance	4.47	1.38	5.01

	Link 1, 2	Link 1, 3	Link 2, 3
Covariance	1.81	3.55	2.50
Correlation coefficie	0.73	0.75	0.95

and variance of Link 3 are greater than Link 1 and Link 2 although mean flow on Link 3 is the least. We can also calculate covariance between link travel times. Thus, we can evaluate network's uncertainty using the model presented.

CONCLUSION

Evaluating uncertainty of traffic networks is very important for network design or traffic management. We assume that drivers choose their routes stochastically based on the logit model and that the travel demands are negative-binomial-distributed. A network equilibrium model with stochastic route choice under stochastic demands is formulated as a fixed point problem and a complementary problem. Then, the model is applied to a simple example. As a future work, we have to examine the properties of the equilibrium under gamma-distributed demand are discussed. Also, for applying for a large-scale network, an algorithm should be developed.

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