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Wardrop's equilibrium assignment scheme without traffic volume for estimating link travel time

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1. Introduction

Traffic assignment is one of the methods to analyse the degree of congestion in the network. Especially, equilibrium assignment techniques, which have been firstly introduced by Wardrop (1956), forms the main part of traffic assignment problems and has been utilised to analyse many problems of road network systems.

The concept of traffic volume has been considered as a necessary factor in equilibrium assignment problems in road networks. Travel times of links in road networks are calculated from the number of vehicles passing through each link. The number of vehicles in the networks is determined by OD travel demand. Therefore it has been considered that the level of congestion of the network can be known only when the amount of travel demand is known

Although the information of traffic volume seems to be necessary to analyse road networks, there are some difficulties in measuring traffic volume in the real world. The most difficult issue related to traffic volume is "how to determine OD demand volume". Measuring OD demand volume directly is quite difficult because tracking all the vehicles' movement is necessary to know the exact OD demand volume. Tracking the all personal movement can be considered as an impossible manner and therefore we may have to consider that there is no way to know true OD demand volume. One of the theoretical methods to tackle with this issue is the "OD demand volume estimation from link traffic volume". Many methodologies of OD demand volume estimation have been proposed, however, it is known that OD demand volume cannot be determined uniquely from the information of link traffic volume (e.g. Bierlaire, 2002).

A possible way to avoid these difficulties is "eliminating all the variables and constants related to traffic volume from a theory". Though knowing information on traffic volume is demanded in many situations, however, only knowing travel time may be enough in some cases. Information on travel time of each link is considered as the important index indicating congestion or level of service of road networks, for example. This study proposes a theoretical framework of Wardrop's equilibrium assignment without the concept of traffic volume. An equation system describing equilibrium condition is stated first without the concept of traffic volume. Then, mathematical characteristics of the equation system are analysed. As an application of this theory, a methodology to estimate the level of congestion with the limited measurement of link travel time is proposed.

2. Formulation of an equilibrium system without traffic volume variables.

An equilibrium system will be derived from the normal formulation of the Wardrop's equilibrium system. The problem of finding link travel times which satisfies Wardrop's equilibrium condition is described as

$$\begin{aligned}
&\text{Find the set of } \mathbf{t} \text{ satisfying} \\
&c_r \geq \overline{c}_{i(r)}, f_r\left(c_r - \overline{c}_{i(r)}\right) = 0, f_r \geq 0 \quad && \text{for } \forall r \in R \\
&t_l = \phi\left(x_l\right) && \text{for } \forall l \in E \\
&\sum_{r \in R_\Omega(i)} f_r = Q_i && \text{for } \forall i \in \Omega
\end{aligned} \tag{1}$$

where

- E — The set of all links in the network
- V — The set of all nodes in the network
- $\Omega \subset V \times V$ — The set of all OD pairs in the network
- R — The set of all routes connecting each OD pair
- $E_R(r)$ — The set of links belonging to the route r
- $R_\Omega(i)$ — The set of routes belonging to the OD pair i
- $R_E(l)$ — The set of routes passing through the link l
- $i(r)$ — OD pair connected by route r
- t_l — Travel time of link l. Not negative. $\mathbf{t}$ is a vector of t_l.
- $\overline{c}_i$ — Minimum travel time of the OD pair i. Not negative.
- c_r — Travel time of the route r, must be $c_r = \sum_{l \in E_R(r)} t_l$. Not negative.
- f_r — Traffic volume using the route r. Not negative.
- x_l — Link traffic volume of link l, must be $x_l = \sum_{r \in R_E(l)} f_r$. Not negative.
- $\phi(x)$ — Link travel time function, indicating link travel time where traffic volume is x.
- Q_i — OD demand volume of OD pair i. Not negative nor zero.

The equation system (1) includes variables describing travel time and variables describing traffic volume. Assuming that we cannot have information of OD traffic demand volume but can know which OD pairs have nonzero OD demand, the equation system (1) will be replaced by

$$\begin{array}{ll} \text{Find the set of } \mathbf{t} \text{ satisfying} & \\ c_r \geq \overline{c}_{i(r)}, f_r\left(c_r - \overline{c}_{i(r)}\right) = 0, f_r \geq 0 & \text{for } \forall r \in R \\ t_l = \phi\left(x_l\right) & \text{for } \forall l \in E \\ \sum_{r \in R_\Omega(i)} f_r > 0 & \text{for } \forall i \in \Omega \end{array} \tag{2}$$

Only the last condition is changed from (1), meaning that all the cases where all OD pairs belonging to Ω have positive demand are considered in the equation system (2).

Also we have to assume that the exact shape of link travel time function is not known because its argument is link traffic volume. Instead, we can consider a criterion such as

- If traffic exists, travel time is not less than free flow travel time.
- If no traffic exists, travel time must be equal to free flow travel time.

This criterion shows the basic condition on link travel time functions because free flow travel time is defined as the travel time without traffic and minimal travel time is achieved if no other traffic exists on the road. Considering this scheme, the equation system (2) can be

$$\begin{array}{ll} \text{Find the set of } \mathbf{t} \text{ satisfying} & \\ c_r \geq \overline{c}_{i(r)}, f_r\left(c_r - \overline{c}_{i(r)}\right) = 0, f_r \geq 0 & \text{for } \forall r \in R \\ 0 \leq t_l - t_l^0 \leq M_l \theta\left(x_l\right) & \text{for } \forall l \in E \\ \sum_{r \in R_\Omega(i)} f_r > 0 & \text{for } \forall i \in \Omega \end{array} \tag{3}$$

where $\theta\left(x_l\right)$ is a function whose value is 1 only when the argument is positive and is zero when the argument is zero. t_l^0 is free flow link travel time of link l, which is defined on each link externally. M_l is a sufficient large positive number indicating the practical upper bound of link delay. Theoretically the value of M_l would be infinite, but we can set certain big value to M_l (say, link travel time made by a vehicle running at a speed of 0.1 km/hour) instead of infinite in practical problems.

The solution set of (3) can be considered as "a set of possible link travel times". The link travel times $\mathbf{t}$ solved by the normal Wardrop's equilibrium condition (1) can also satisfy (3) because no new condition is added to (3) whereas two conditions, that is, OD demand constraint and link travel time functions, are loosened. Therefore we can find the solution of the equilibrated network described by (1) from the solution set of the equation system (3). Apparently, the set of $\mathbf{t}$ solved by (3) must not empty because the original Wardrop's problem has a solution.

Note that the equation system (3) can be transformed into

$$\begin{array}{ll} \text{Find the set of } \mathbf{t} \text{ satisfying} & \\ c_r \geq \overline{c}_{i(r)}, f_r\left(c_r - \overline{c}_{i(r)}\right) = 0, f_r \geq 0 & \text{for } \forall r \in R \\ 0 \leq t_l - t_l^0 \leq M_l \theta\left(x_l\right) & \text{for } \forall l \in E \\ \sum_{r \in R_\Omega(i)} f_r = 1 & \text{for } \forall i \in \Omega \end{array} \tag{4}$$

where the last inequality, which shows that the OD traffic is greater than zero, is replaced by an equation. Because the magnitude of f_r does not affect the solution set of the equation system, we can set an arbitrary OD traffic demand instead of using the inequality.

3. <u>Condition for obtaining an upper bounded set of possible link travel times</u>

Though a set of possible link travel times can be obtained with no information on traffic volume, the possibility described by this set is too large. Especially, we have the chance to obtain the solution set where travel time of some link is as big as the practical upper bound of link delay M_l. This situation is not good when we have to evaluate the "worst" case of the network congestion because the evaluation result will depend on the external value of M_l, which cannot be determined easily like the free flow link travel time. Therefore adding some external condition to the equation system (4) must be considered to obtain the upper bounded set of possible link travel times.

Two conditions in the equation system (4) have a chance to determine the upper limit of the link travel time. One is the condition that "if no traffic is loaded on a link, this link must have free flow travel time". The other is "travel times of the routes used by travellers must not greater than travel times of all other routes". These two conditions imply that "if every OD has at least one route whose travel time is known and finite, travel times of all routes are upper bounded and therefore all link travel times are upper bounded". This can be described as

$$\begin{aligned} &t_l \le \hat{c}_i \quad \text{if } l \in E_R(r), f_r > 0 \text{ and } r \in R_\Omega(i) \text{ for } \forall i \in \Omega \\ &t_l = t_l^0 \quad \text{if } x_l = 0 \end{aligned} \tag{5}$$

where $\hat{c}_i = \min_{r \in S_\Omega(i)} c_r$ ($S_\Omega(i)$ is the set of routes of OD pair i whose travel time is known and finite). Note that all the links used by travellers must be included by the first condition of (5), meaning that their travel times are upper bounded. All other links, which must not be used by travellers, should have free flow travel time. Therefore all link travel times are upper bounded in this case.

This condition for obtaining upper bounded travel times implies that we may have to measure travel time of "some" part of links. In other words, however, this result shows that we do not have to measure "whole" links of the network to know the degree of congestion of the network. For example, if floating cars traverses all the origins and destinations, we can estimate the degree of congestion without measuring all links in the network. Such advantage can be utilised to measure the degree of congestion of the complex network with lower cost. Especially, if the degree of congestion of many cities must be evaluated at once to have statistics of road congestion, this technique could be convenient. There are some countries like UK which collects such statistics by floating vehicle method (Department for Transport, 2004), however it only evaluates travel times of limited links where measurement is made. The theory shown in this study can evaluate congestion of "whole network" with the measurement on limited links.

4. Optimisation problem for evaluating network congestion and its solution method

Though the set of possible travel times can be known by analysing the equation system (4), the shape of this set can be quite complicated. Determining "congestion indicator", which is a function of travel times of all links (total travel time of all links or some specified links, for example), can be good way to avoid this problem. Formulating an optimisation problem whose optimal function is a congestion indicator and whose feasible area is determined by the equation system (4), the upper bound and lower bound of the congestion indicator can be obtained. However, it must be noted that the equation system (4) have route variables such as f_r, meaning that enumeration all the routes is inevitable, which can take huge calculation cost.

A node-link formulation (Patriksson, 1994) is introduced to avoid route enumeration.

Using the node-link formulation of equation system (4) as a feasible set of the optimisation problem, we obtain

$$
\begin{aligned}
&\text{Minimise or Maximise } \Phi(\mathbf{t}) \text{ subject to} \\
&x_{pq}^{i}\left(t_{pq}-\Delta\pi_{pq}^{i}\right)=0,\ t_{pq}\geq\Delta\pi_{pq}^{i} \text{ and } x_{pq}^{i}\geq 0 \quad \text{for } \forall(p,q)\in E, \forall i\in\Omega && \text{(Cond. A)} \\
&\sum_{q\in D_p} x_{pq}^{i}-\sum_{q\in U_p} x_{qp}^{i}=\begin{cases}1 & \text{if } p=o(i)\\ \text{-1} & \text{if } p=d(i)\\ 0 & \text{otherwise}\end{cases} \quad \text{for } \forall p\in V, \forall i\in\Omega && \text{(Cond. B)} \quad (6) \\
&0\leq t_{pq}-t_{pq}^{0}\leq M_{pq}\theta\left(\sum_{i\in\Omega} x_{pq}^{i}\right) \quad \text{for } \forall(p,q)\in E && \text{(Cond. C)} \\
&t_{pq}=t_{pq}^{MEAS} \quad \text{for } \forall(p,q)\in E^{MEAS} && \text{(Cond. D)}
\end{aligned}
$$

where

- $\Phi(\mathbf{t})$ Congestion indicator of the network
- $o(i)$ Origin node of OD pair i
- $d(i)$ Destination node of OD pair i
- (p,q) Link stating from the node p and terminated by the node q
- x_{pq}^{i} Link traffic volume contributed by vehicles belonging to OD pair i
- π_{p}^{i} Node cost (Minimal travel cost from $o(i)$ to node p)
- $\Delta\pi_{pq}^{i}$ Difference of node cost, defined as $\Delta\pi_{pq}^{i}=\pi_{q}^{i}-\pi_{p}^{i}$
- U_p Upstream nodes set: defined as $U_p=\{q|q\in V,(q,p)\in E\}$
- D_p Downstream nodes set: defined as $D_p=\{q|q\in V,(p,q)\in E\}$.
- t_{pq}^{MEAS} Measured travel time: Travel time of links where measurement is made
- E^{MEAS} Set of measured links.

The last condition (Cond. D) of (6) is added to evaluate congestion with travel time measurements made on limited links whose set is denoted by E^{MEAS}.

As stated by Patriksson (1994), the conditions A, B, and C of the problem (6) is equivalent if the route set R contains any cyclic flows. Note that, due to the constraint of equilibrium, no cyclic routes are used in the solution satisfying the constraint of equilibrium. Therefore the conditions A, B, and C of (6) is the same as the solution set of (4) without cyclic flow.

Now the point is that the complementarity condition in the (Cond. A) of (6) is not affected by the magnitude of x_{pq}^{i} but only depends on the sign of x_{pq}^{i}. Thus, the

(Cond. A) of (6) can be replaced by

$$\left.\begin{array}{l} y_{pq}^{i}=\theta\left(x_{pq}^{i}\right) \\ y_{pq}^{i}\left(t_{pq}-\Delta\pi_{pq}^{i}\right)=0 \\ t_{pq}-\Delta\pi_{pq}^{i}\geq 0 \\ x_{pq}^{i}\geq 0 \end{array}\right\} \text{ for } \forall\left(p,q\right)\in E, \forall i\in\Omega \tag{7}$$

This equation (7) can be replaced by the form without complementarity condition such as

$$\left.\begin{array}{l} y_{pq}^{i}=\theta\left(x_{pq}^{i}\right) \\ t_{pq}-\Delta\pi_{pq}^{i}\leq M^{*}\left(1-y_{pq}^{i}\right) \\ t_{pq}-\Delta\pi_{pq}^{i}\geq 0 \\ x_{pq}^{i}\geq 0 \end{array}\right\} \text{ for } \forall\left(p,q\right)\in E,\ \forall i\in\Omega, \tag{8}$$

where $M^{*}\gg M_{pq}$. Though (8) does not contain the complementarity condition, it still has nonlinear and non-smooth function $\theta\left(x\right)$. To eliminate this function, an alternative formulation

$$\left.\begin{array}{l} y_{pq}^{i}=\left\{0,1\right\} \\ x_{pq}^{i}\leq y_{pq}^{i}\leq R^{MAX}x_{pq}^{i} \\ t_{pq}-\Delta\pi_{pq}^{i}\leq M^{*}\left(1-y_{pq}^{i}\right) \\ t_{pq}-\Delta\pi_{pq}^{i}\geq 0 \\ x_{pq}^{i}\geq 0 \end{array}\right\} \text{ for } \forall\left(p,q\right)\in E,\ \forall i\in\Omega, \tag{9}$$

where $R^{MAX}\gg\left|R\right|$, will be examined. In this form, the relationship of $y_{pq}^{i}=\theta\left(x_{pq}^{i}\right)$ is replaced by

$$y_{pq}^{i}=\left\{0,1\right\},\ x_{pq}^{i}\leq y_{pq}^{i}\leq R^{MAX}x_{pq}^{i}. \tag{10}$$

If $x_{pq}^{i}=0$, $y_{pq}^{i}=0$ must be satisfied, meaning that the relationship defined by $y_{pq}^{i}=\theta\left(x_{pq}^{i}\right)$ in the original equation (7) holds in (10). If $1/R^{MAX}\leq x_{pq}^{i}\leq 1$, $y_{pq}^{i}=1$ must be satisfied and $y_{pq}^{i}=\theta\left(x_{pq}^{i}\right)$ also holds. The case where $x_{pq}^{i}>1$ do not have to be considered because OD traffic volume is fixed to 1 and cyclic flow is not realised in equilibrium. The case where $0<x_{pq}^{i}<1/R^{MAX}$ is not allowed in the condition (10) because this is equivalent to the condition $0 < y_{pq}^i < 1$ and $y_{pq}^i = \{0,1\}$, which cannot be satisfied simultaneously. So the condition (10) is equivalent to the condition such as

$$\begin{aligned} & y_{pq}^i = \theta\left(x_{ipq}\right) \\ & x_{pq}^i = 0 \text{ or } x_{pq}^i \geq 1/R^{MAX} \end{aligned} \tag{11}$$

meaning that the equation system (9) is identical to (8) other than the additional constraint on x_{pq}^i. However, despite such additional condition, the equation system (9) can be used an alternative form of (8) because $x_{ipq} \geq 1/R^{MAX}$ can hold for any links and any OD pairs with splitting OD traffic volume into all possible non-cyclic routes equally. Again, note that only the signs of x_{pq}^i affect the equation (8), whereas the exact magnitudes of x_{pq}^i have no effect on the result.

The (Cond. C) of (6), where $\theta(x)$ also appears, can be transformed into

$$0 \leq t_{pq} - t_{pq}^0 \leq M_{pq} \sum_{i \in \Omega} y_{pq}^i , \tag{12}$$

if it can be assumed that the travel time of all links will be less than M_{pq}, where the maximum bound condition of link travel times do not work. Because the link travel times can be less than M_{pq} by satisfying the condition shown in the section 3, the replacement of (12) can be used in most practical situations.

Combining all modifications shown above, the original problem (6) can be transformed into

$$\begin{array}{lll}
\text{Minimise or Maximise } \Phi(\mathbf{t}) \text{ subject to} & & \\
\left.\begin{array}{l} x_{pq}^i \geq 0,\ y_{pq}^i = \{0,1\},\ x_{pq}^i \leq y_{pq}^i \leq R^{MAX} x_{pq}^i \\ t_{pq} - \Delta\pi_{pq}^i \leq M^*\left(1 - y_{pq}^i\right),\quad t_{pq} \geq \Delta\pi_{pq}^i \end{array}\right\} & \text{for } \forall(p,q) \in E, \forall i \in \Omega & \text{(Cond. A}'\text{)} \\
\sum_{q \in D_p} x_{pq}^i - \sum_{q \in U_p} x_{qp}^i = \begin{cases} 1 & \text{if } p = o(i) \\ \text{-1} & \text{if } p = d(i) \\ 0 & \text{otherwise} \end{cases} & \text{for } \forall p \in V, \forall i \in \Omega & \text{(Cond. B)} \\
0 \leq t_{pq} - t_{pq}^0 \leq M_{pq} \sum_{i \in \Omega} y_{pq}^i & \text{for } \forall(p,q) \in E & \text{(Cond. C}'\text{)} \\
t_{pq} = t_{pq}^{MEAS} & \text{for } \forall(p,q) \in E^{MEAS} & \text{(Cond. D)}
\end{array} \tag{13}$$

Note that this modified optimisation problem (13) is a zero-one mixed integer linear programming if $\Phi(\mathbf{t})$ is linear, which can be solved by the branch and bound method.

5. <u>A numerical test</u>

A numerical test is carried out in the small network shown in figure 1. This network has six nodes and 14 links. Free flow travel times of all links are 5 minutes. Four nodes are OD nodes, where positive OD demand is assumed between any combinations of these nodes. Travel times of nine links are known. Delay is detected only on the link (3,2), which is shown by thick arrow. Total delay of all links, which is defined as the difference between total travel time and total free flow travel time, is employed as the optimisation function $\Phi(\mathbf{t})$. Both the maximal and minimal number of $\Phi(\mathbf{t})$ are calculated.

The result of maximal travel time is shown in figure 2 and the result of minimal travel time is shown in figure 3. Total delays of both cases are calculated as 120 minuets and 30 minutes respectively. Figures 2 and 3 shows travel time of each link in the solutions , however, it must be noted that these values may not be determined uniquely and they can be only one of a solution set which can perform optimal solutions.

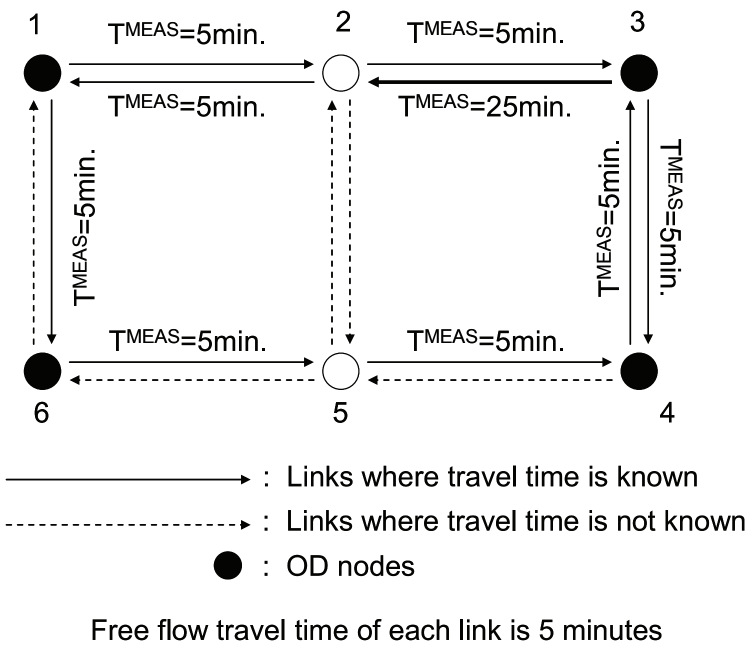

Figure 1: A network for numerical test

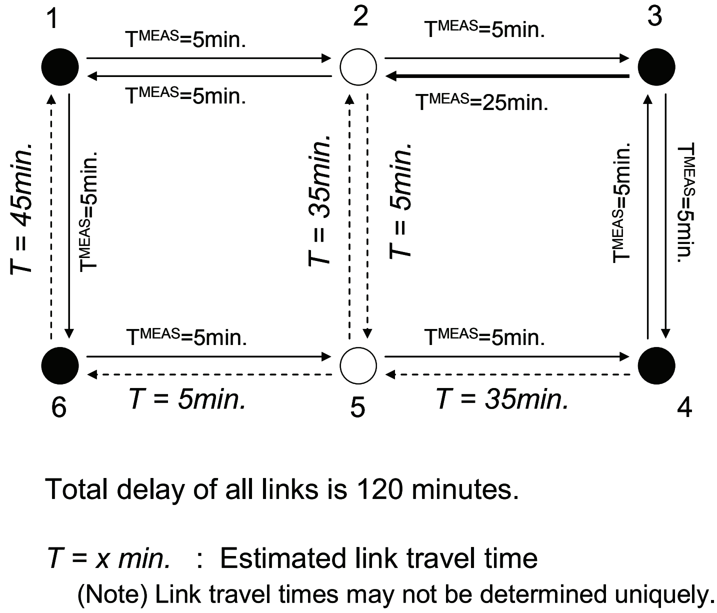

Figure 2: A result of total link delay maximisation problem

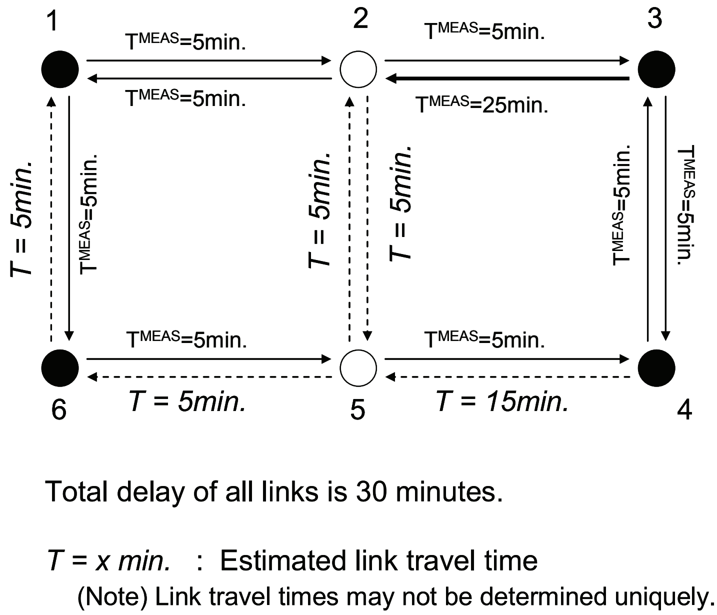

Figure 3: A result of total link delay minimisation problem

6. Conclusions and discussions

This study proposes a Wardrop's equilibrium assignment problem without traffic volume and shows an application of the scheme to estimating congestion in network. This problem provides a set of link travel times which can be feasible in equilibrated network. Though the size of this set can be very large, we can reduce the set of size by assuming travel times of some links. This study also shows that assuming travel time of at least one route per OD pair (note that this route does not have to be actually used) is sufficient to limit the size of the set.

This study proposes an optimisation problem for estimating congestion in network with measuring travel times of limited links. The original problem including a non-linear function, route enumeration, and a complementarity condition has been transformed into a zero-one mixed integer linear programming. A numerical example has been shown.

This study shows a new concept of "non-traffic-based equilibrium assignment scheme". The framework shown in this study is a simplest case of non-traffic-based equilibrium problem whose assumption is based on the original Wardrop's concept. There can be many expansions of the non-traffic-based equilibrium problem based on a dynamic user equilibriums scheme, a stochastic user equilibrium scheme, or others.

This methodology can be utilised to estimate traffic congestion with limited measurement. Especially, this methodology may be quite useful when a survey of traffic congestion in many cities where road network is complicated is planned. The importance of congestion survey in road networks has been often pointed out (Bovy and Salomon, 2002, for example) and developing a methodology for estimating congestion would help to achieve practical objectives.

The methodology of this study, however, still has many problems in practical aspects. One of the most important issues is that the difference between the maximal estimation and the minimal estimation can be big. As shown in the section 5, for example, the maximal estimation can be several times of the minimal estimation. Tackling with these problems should be needed to apply the proposed method to empirical issues.

Acknowledgements

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