Routing Optimization with Deadlines under Uncertainty - Theory and Computational Results

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1 Problem Description

We study routing problems through networks with deadlines imposed at a subset of nodes, and uncertain arc travel times characterized by a distributional information set incorporating ambiguity. Our model is static in the sense that routing decisions are made prior to the realization of uncertain travel times.

To incorporate ambiguity, instead of defining an exact probability distribution $\mathbf{P}$ for an arc travel time, we assume its true distribution lies in a family of distributions denoted by $\mathbf{F}$, which is characterized by some descriptive statistics, e.g., mean and bound support. The goal is to find optimal routing policies such that arrival times at nodes respect deadlines "as much as possible", in a mathematically precise way under an appropriately defined performance measure which takes into account such distributional uncertainty assumptions.

Our model can be applied to transportation networks, for example, for delivery service providers to route their vehicles, where multiple vehicles and uncertain service time could be incorporated, or for an individual to make his/her travel plan.

2 Models and Solutions

We consider a directed strongly connected network $G = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N} = \{1, \ldots, n\}$ is the set of nodes, nodes 1 and n representing an origin and a destination, respectively, and $\mathcal{A}$ is the set of arcs. The set $\mathcal{I} \subseteq \mathcal{N}$ represents the nodes with deadline requirement. Without loss of generality, we assume that node $n \in \mathcal{I}$. We use τ_i to represent the deadline or target time prespecified at node $i \in \mathcal{I}$ (an input), and $\tilde{t}_i$ to be a random variable, denoting the real arrival time at node $i \in \mathcal{N}$, from a specific routing policy.

2.1 Stochastic shortest path problem with deadline

Here we consider a special case where we have only one node with a deadline, node n, i.e. $\mathcal{I} = \{n\}$. We first define the worst-case certainty equivalent (WCE) arrival time as a deterministic amount of travel time that an ambiguity-averse individual would view as equally acceptable as the uncertain arrival time $\tilde{t}_n$ under an exponential disutility with a given risk tolerance parameter $\alpha > 0$, i.e.,

$$C_{\alpha,\mathbb{F}}\left(\tilde{t}_n\right) = \sup_{\mathbb{P}\in\mathbb{F}} \alpha \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(\frac{\tilde{t}_n}{\alpha}\right)\right).$$

Our main proposed performance measure, called **lateness index**, represents the minimum risk tolerance parameter α such that the WCE arrival time is no larger than its deadline:

$$\rho_{\tau_n}\left(\tilde{t}_n\right) = \inf\left\{\alpha \geq 0 \,\middle|\, C_{\alpha,\mathbb{F}}\left(\tilde{t}_n\right) \leq \tau_n\right\}.$$

The problem is to find a path from 1 to n that minimize the lateness index at n.

2.2 Stochastic routing problem with deadlines

In order to consider the whole network performance and account for the inherent relationships between arrival times at nodes, we extend the lateness index to the overall network. Let $\boldsymbol{\tau} = (\tau_i)_{i\in\mathcal{I}}$, $\tilde{\boldsymbol{t}} = \left(\tilde{t}_i\right)_{i\in\mathcal{I}}$, and $\boldsymbol{\alpha} = (\alpha_i)_{i\in\mathcal{I}}$. The **composite lateness index** for the overall network is defined as $\rho_{\boldsymbol{\tau}}\left(\tilde{\boldsymbol{t}}\right) : \mathcal{V}^{|\mathcal{I}|} \to [0, +\infty)$:

$$\rho_{\boldsymbol{\tau}}\left(\tilde{\boldsymbol{t}}\right) = \inf\left\{\varphi(\boldsymbol{\alpha}) \mid C_{\alpha_i,\mathbb{F}}\left(\tilde{t}_i\right) \leq \tau_i, \alpha_i \geq 0, \forall\, i \in \mathcal{I}\right\},$$

in which the function $\varphi(\boldsymbol{\alpha})$ is non-decreasing and convex in $\boldsymbol{\alpha} \geq \mathbf{0}$, with boundary conditions $\varphi(\mathbf{0}) = 0$, and $\lim_{\alpha_j \to +\infty} \varphi((\alpha_1, \cdots, \alpha_j, \cdots, \alpha_{|\mathcal{I}|})') = +\infty$ for all $j \in \mathcal{I}$.

The problem is to find a route from 1 to n going through each node with deadlines in such a way as to minimize the composite lateness index.

2.3 Solution procedures

We can formulate our stochastic routing problem with deadlines as a mixed integer nonlinear optimization problem. For the special case of the stochastic shortest path problem and under the assumption that arc travel times are independent, we prove that the problem is polynomially solvable. We formulate the general problem using a multi-commodity flow formulation and solve it using Benders' decomposition, in which the Lagrangian relaxation technique is used to calculate the sub-gradient for the master problem, and classical convex algorithm is employed to solve the subproblem.

2.4 Computational results

We first conduct a comparative study on the stochastic shortest path problem. We construct a randomly generated network with 300 nodes, around 1500 arcs for each instance, and find the optimal paths under several selection criteria: minimizing average arrival time, maximizing arrival probability, maximizing punctual ratio, and maximizing price of robustness (Bertsimas and Sim, 2003). With the out-of-sample simulation, we conclude that among 50 instances we generate, our lateness index model performs relatively well in terms of lateness probability, expected lateness, value at risk, and computation time.

We also provide a computational study on the general routing optimization problem, which in total has $|\mathcal{A}||\mathcal{I}| + |\mathcal{A}| + |\mathcal{I}|$ binary variables, and $O(|\mathcal{I}||\mathcal{A}|)$ constraints. The program is coded in python and run on a Intel Core i7 PC with a 3.40 GHz CPU by calling CPLEX 12 as ILP solver. We summarize the computation time by varying three key factors: $|\mathcal{N}|$, $|\mathcal{A}|$, and $|\mathcal{I}|$. Table 1 provides the statistics of computation time for each setting with randomly generated 50 instances.

$G = \{\mathcal{N}, \mathcal{A}\}$	$\lvert\mathcal{I}\rvert = \lvert\mathcal{N}\rvert$				$\lvert\mathcal{I}\rvert = 2$			
	Average	Max	Min	STDEV	Average	Max	Min	STDEV
$\lvert\mathcal{N}\rvert = 10$, $\lvert\mathcal{A}\rvert = 30$	0.38	1.18	0.18	0.20	0.13	0.30	0.06	0.05
$\lvert\mathcal{N}\rvert = 10$, $\lvert\mathcal{A}\rvert = 50$	5.53	40.21	0.48	7.25	0.19	0.77	0.07	0.11
$\lvert\mathcal{N}\rvert = 20$, $\lvert\mathcal{A}\rvert = 60$	3.74	15.25	1.14	2.72	0.30	0.60	0.22	0.07
$\lvert\mathcal{N}\rvert = 30$, $\lvert\mathcal{A}\rvert = 90$	28.89	197.23	3.01	37.96	0.64	1.06	0.47	0.11
$\lvert\mathcal{N}\rvert = 40$, $\lvert\mathcal{A}\rvert = 120$	481.14	4250.57	9.62	827.38	0.42	3.23	0.15	0.46

Table 1: Computation time (sec) on routing optimization problem with different settings.

When the number of deadline nodes, i.e., $|\mathcal{I}|$ is relatively small, our algorithm performs quite efficiently. In addition, by setting the computation time limit as 2 hours, our algorithm could solve a network with 60 nodes, 180 arcs when $\mathcal{I} = \mathcal{N}$, and a network with 400 nodes, 2000 arcs when $|\mathcal{I}| = 2$.

3 Related Work

The stochastic variant of routing problem has attracted increased attention due to the intrinsic uncertainties arising in real-world problems. However, only few studies consider the routing problem with deadlines in the presence of uncertain travel time.

Laporte et al. (1992) consider a multiple vehicle routing problem with uncertain travel time and service time. Each vehicle has a targeted time to complete the route. On the premise of additive probability distributions of both travel and service times, they propose a chance constrained model by ensuring the probability of a deadline violation is less than a threshold, and a stochastic programming model by penalizing that lateness.

Kenyon and Morton (2003) mainly focus on the length of the longest route traveled by multiple vehicles and develop two versions of the model by minimizing the expected completion time or maximizing the probability of completion within a given deadline, and finally solve the model by branch-and-cut scheme.

Chang et al. (2009), with the normal distribution assumption, investigate the stochastic routing problem with time windows by guaranteeing the probability of violating the demanded latest time is no larger than a threshold. Russel and Urban (2008) study the problem with time windows by assuming the travel time follows a shifted gamma distribution. They minimize some functions of penalty incurred from the deviation of the time window, and develop tabu-search metaheuristic to solve the problem.

To achieve robust performances, Montemanni et al. (2007) formulate the travel times as a range of possible values, and solve the problem by minimizing the robust deviation.

Finally, we refer the readers to Brown and Sim (2009) for the conceptual framework of satisficing measure in the context of target-based decision making.

References

D.B. Brown, and M. Sim, "Satisficing measures for analysis of risky positions", *Management Science* 55, 71-84, 2009.

T.S. Chang, Y.W. Wan, and W.T. OOI, "A stochastic dynamic traveling salesman problem with hard time windows", *Eur. J. of Oper. Res.* 198, 748-759, 2009.

A.S. Kenyon, and D.P. Morton, "Stochastic vehicle routing with random travel times", *Transportation Science* 37, 69-82, 2003.

G. Laporte, F. Louveaux, and H. Mercure, "The vehicle routing problem with stochastic travel times", *Transportation Science* 26 (3), 161-70, 1992.

R. Montemanni, J. Barta, M. Mastrolilli, and L.M. Gambardella, "The robust traveling salesman problem with interval data", *Transportation Science* 41(3), 366-381, 2007.

R.A. Russell, and T.L. Urban, "Vehicle routing with soft time windows and Erlang travel times", *Journal of the Operational Research Society* 59, 1220-1228, 2008.

D. Bertsimas, and M. Sim, "Robust discrete optimization and network flows", *Mathematical Programming* 98, 4971, 2003.