Branch-price-and-cut for inventory routing under a maximum level replenishment policy

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The inventory routing problem (IRP) is an important problem faced by distribution companies that manage the inventory of their customers. Given one supplier producing a known quantity of a single product at each period of a finite planning horizon, a set of customers consuming a known quantity of this product at each period and a homogeneous fleet of capacitated vehicles owned by the supplier, the IRP consists of building feasible delivery vehicle routes in each period such that no stockouts occur at the customers while respecting inventory capacity at each of them. The objective is to minimize the sum of the vehicle traveling costs and the inventory holding costs at the supplier and at the customers. We consider a maximum level replenishment policy that allows to deliver any quantity to a customer as long as the inventory does not exceed its maximum level. We assume the following sequences of operations. At each period, the supplier performs production before making deliveries. Customers receive their deliveries at the beginning of the periods and can use them to fulfill their demands right away. Holding costs are charged on the inventory at the end of a period.

Various variants of the IRP have been studied in the literature but only a few exact algorithms were developed for the basic IRP stated above. Branch-and-cut algorithms for the single-vehicle case were proposed by Archetti et al. (2007) and Solyalı and Süral (2011). For the multi-vehicle case, Coelho and Laporte (2013) introduced a unified model for several IRP variants, Archetti et al. (2012) proposed several models for the basic IRP, and Adulyasak et al. (2013) presented different models for the production routing problem that includes the IRP as a special case. All these authors developed branch-and-cut algorithms for solving their models.

The main contribution of this paper is the development of the first exact branch-price-and-cut algorithm for the multi-vehicle IRP under a maximum level replenishment policy. In particular, we introduce a new formulation and several families of valid inequalities.

1 A path-flow model

Consider the following notation. $P = \{1, 2, \ldots, \rho\}$: set of periods; $\rho + 1$: fictitious period used for customer end inventories; N: set of customers; $N_0 = N \cup \{0\}$: set of locations where 0 represents the supplier; C_i: inventory holding capacity at customer i; h_0: holding cost at the supplier; d_0^p: quantity produced by the supplier in period p; d_i^p: demand at customer i in period p that cannot be satisfied by its initial inventory; I_0^0: initial inventory at the supplier; P_{ip}: subset of periods in which a delivery can be performed at customer i to fulfill its demand at period p or stock its end inventory if $p = \rho + 1$; R: set of feasible routes (paths); a_{ri}: binary parameter equal to 1 if customer i is visited in route r; W_r^p: set of feasible extreme delivery patterns for route r if operated in period p (a pattern indicates the following quantities); q_{wi}^s: quantity delivered in pattern $w \in W_r^p$ to satisfy the demand of customer i in period s or to stock its end inventory if $s = \rho + 1$; q_w: total quantity delivered in pattern w; c_{rw}: cost of route r with delivery pattern w that includes travel costs and inventory costs yielded by the deliveries; and K: number of vehicles available.

Two types of variables are used: binary variable y_{rw}^p takes value 1 if and only if route r is operated with delivery pattern w in period p; nonnegative variable I_0^p indicates the inventory at the supplier at the end of period p.

Using this notation, we can formulate the IRP as the following mixed integer program:

$$\min \quad \sum_{p\in P}\sum_{r\in R}\sum_{w\in W_r^p} c_{rw} y_{rw}^p + \sum_{p\in P} h_0 I_0^p \tag{1}$$

$$\text{s.t.} \quad I_0^{p-1} + d_0^p - \sum_{r\in R}\sum_{w\in W_r^p} q_w y_{rw}^p = I_0^p, \qquad \forall p \in P, \tag{2}$$

$$\sum_{p\in P_{is}}\sum_{r\in R}\sum_{w\in W_r^p} q_{wi}^s y_{rw}^p = d_i^s, \qquad \forall i \in N, s \in P, \tag{3}$$

$$\sum_{p\in P_{i,\rho+1}}\sum_{r\in R}\sum_{w\in W_r^p} q_{wi}^{\rho+1} y_{rw}^p \le C_i, \qquad \forall i \in N, \tag{4}$$

$$\sum_{r\in R}\sum_{w\in W_r^p} a_{ri} y_{rw}^p \le 1, \qquad \forall i \in N, p \in P, \tag{5}$$

$$\sum_{r\in R}\sum_{w\in W_r^p} y_{rw}^p \le K, \qquad \forall i \in N, p \in P, \tag{6}$$

$$I_0^p, y_{rw}^p \ge 0, \qquad \forall r \in R, w \in W_r, p \in P, \tag{7}$$

$$\sum_{w\in W_r^p} y_{rw}^p \in \{0,1\}, \qquad \forall r \in R, p \in P. \tag{8}$$

Objective function (1) minimizes the total traveling and holding costs. Constraints (2) balance the inventory at the supplier from one period to the next. Constraints (3) and (4) ensure that the demand of each customer is met in each period and that its end inventory does not exceed its holding capacity. Next, constraints (5) and (6) ensure that, in each period, every customer is visited at most once and no more than K vehicles are used. Nonnegativity requirements on the variables are given by (7). Binary requirements are not imposed directly on the y^p_{rw} variables, but rather on the routes themselves, allowing convex combinations of feasible extreme delivery patterns (as in Desaulniers (2010) for the split delivery vehicle routing problem with time windows).

2 Branch-price-and-cut algorithm

To solve model (1)–(8), we propose a branch-price-and-cut algorithm, that is, column generation is applied to compute lower bounds, cutting planes are generated to tighten these bounds, and if needed branching decisions are imposed to derive integer solutions. In the column generation algorithm, the master problem consists of the linear relaxation (1)–(7) and there is one subproblem per period. Similar to the subproblem in Desaulniers (2010), it corresponds to an elementary shortest path problem with resource constraints combined with a bounded knapsack problem. To ease its solution by a labeling algorithm, we rather use the ng-path relaxation introduced by Baldacci et al. (2011) that allows certain cycles according to neighborhoods defined for every customer.

We consider several families of valid inequalities. First, for each customer, we consider inequalities that impose a minimum number of deliveries to meet the total demand between the beginning of the horizon and each period. This minimum number of deliveries is computed based on the demand in each period, the inventory capacity at the customer and vehicle capacity. The second family of inequalities is as follows. For each customer $i \in N$ and period $s \in P$ such that $d^s_i > 0$, the valid inequality is:

$$\sum_{p \in P_{is}} \sum_{r \in R} \sum_{w \in W^p_r} a_{ri}(1 + b^s_{wi}) y^p_{rw} \geq 2, \tag{9}$$

where b^s_{wi} is a binary parameter equal to 1 if $q^s_{wi} = d^s_i$ and 0 otherwise. These inequalities can be generalized for subsets of consecutive periods $s \in P$. Finally, we also apply well-known rounded capacity cuts that are deduced from an auxiliary network in which the nodes correspond to pairs of customer i and period s such that $d^s_i > 0$.

Noticing that traversing a route is equivalent to traversing the same route in the reverse direction, one can prove that the integrality requirements (8) can be replaced by integrality requirements on the edge flows in the networks underlying the problem. Therefore, to reduce symmetry, we propose branching decisions on the edge flows together with decisions on the number of vehicles used in each period and on the number of visit

(0 or 1) to each customer in each period.

3 Computational results

At the moment, the branch-and-cut algorithms of Coelho and Laporte (2013) and Adulyasak et al. (2013) are the two best algorithms for solving the IRP. Given that we have access to the complete results of the algorithm of Coelho and Laporte (2013) (ran on a single thread), we compare our results to theirs. Our preliminary computational results on the Archetti et al. (2007) instances adapted to the multi-vehicle case (involving 2 to 5 vehicles) show that the linear relaxtion (1)–(7) is much tighter than that of the arc-flow model of Coelho and Laporte (2013). However, given the time required to solve this linear relaxation in the 2- and 3-vehicle cases, our algorithm is not yet competitive with their algorithm. On the other hand, for several instances with 4 and 5 vehicles, our branch-price-and-cut algorithm outperforms the branch-and-cut algorithm of Coelho and Laporte (2013).

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