System optimum formulation for departure time choice problem in the generalized bathtub model

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1 INTRODUCTION

Traffic congestion occurs when the traffic density increases while the traffic flow remains constant or decreases. Macroscopic models aim to rule out urban traffic congestion by holding several assumptions. The common assumption between all macroscopic models is the homogeneity of the speed at a single zone with respect to its traffic density (i.e., accumulation). Seminal work of Geroliminis & Daganzo (2008) showed that single-unit macroscopic models could represent the congestion of an urban area independently of the demand. In order to better understand the congestion, accurate calculation of traffic network optimums are crucial. Network operators aim to minimize the sum of all user costs. The solution is called system optimum (SO) (Wardrop, 1952). To accurately calculate SO at the macroscopic level, we need to consider the characteristics of trips. In the context of macroscopic models, a given trip has limited attributes. In simplified models, e.g., bottleneck models (Li *et al.*, 2020), a trip is only defined by its departure time and desired arrival time. However, the characteristics of the user trip (path choice) have to be considered. Macroscopic fundamental diagram (MFD)/bathtub models are able to overcome this drawback by considering trip-length distribution in addition to departure time and desired arrival time distributions. Classic bathtub model Vickrey (1991) assumes that time-independent negative exponential distribution represents the remaining trip distance of all trips traveling in the system. This assumption is cannot represent a real test case based on the empirical studies (cf. e.g. Liu *et al.* (2012)). Recently, Jin (2020) proposed the generalized bathtub model that extends the classic bathtub model to capture various distributions of the trip length by introducing a new state variable: the number of active trips at time t with remaining distances greater than or equal to threshold x, denoted by $K(x, t)$. He formulates the traffic dynamics by four equivalent partial differential equations that track the distribution of the remaining trip lengths (Jin, 2020). This study aims to formulate and solve system optimum, also known as social optimum, for departure time choice model based on generalized bathtub model with heterogeneous trip attributes. In particular, the proposed framework is able to address any distribution for desired arrival time and trip length.

2 METHODOLOGY

The notations are collected in table 1. Bathtub models assume that at time t, the velocity (v_t) is the same for all traveling users. v_t is a function of the network characteristics and the network load, that is to say, the number of travellers in the network, $H(t)$. Let us define the characteristic travel distance $z(t)$ as the distance traveled by a virtual user up until time t:

$$z(t) := \int_0^t v_s ds. \tag{1}$$

Where $v_t = V(H(t))$, and V is assumed bounded from above and below: $0 < V_{min} \leq V \leq V_{max}$. Since $v_t > 0\ \forall t \in \mathcal{T}$, z is an invertible function. Let z^{-1} denote the inverse function of z. Then, we have $z^{-1}\big(z(t)\big) = t$ and $z^{-1}(x)$ represents the time at which the virtual user has reached x. Note that the negative exponential distribution of the trip length transforms the generalized bathtub model to the classic bathtub model (Jin, 2020). Therefore, the results from both models will be identical. It is worth mentioning that the assumption of exponential distribution for the demand profile also transforms other common macroscopic models (e.g., MFD or trip-based MFD models) to the simple accumulation model and results in the same solution (see Lamotte *et al.* 2018 for the details).

Now, let $T(t_d^i, x^i)$ denote the travel time of a player departing at time t_d^i with trip length x^i. Considering (1), $T(t_d^i, x^i)$ can be determined by,

$$T(t_d^i, x^i) = z^{-1}\big(x^i + z(t_d^i)\big) - t_d^i. \tag{2}$$

In assignment problems, the travel cost is usually defined based on α-β-γ scheduling preferences (Fosgerau, 2015). That means the cost function is defined as the sum of the travel time and a penalty cost for arriving at $t_d^i + T(t_d^i, x^i)$ instead of the desired arrival time. Specifically, we assume that each player's cost function is given by,

$$J_i(t_d^i, t_a^i; t_d^{-i}, x^{-i}) = \alpha T(t_d^i, x^i) + \beta\big(t_a^i - t_d^i - T(t_d^i, x^i)\big)_+ + \gamma\big(t_d^i + T(t_d^i, x^i) - t_a^i\big)_+, \tag{3}$$

where α denotes the cost of traveling per unit of time, β and γ denote, respectively, the cost of earliness and lateness for the traveller arrival. We assume that the penalty for travel time is higher than the penalty for early arrival time, i.e., $\alpha > \beta$. Note that $(y)_+ = \max\{y, 0\}$ as well t_d^{-i} and x^{-i} respectively express the dependency of J on the departure times and trip lengths of the other users ($\neq i$) via their travel times.

The cost function defined in (3) captures the fact that travellers prefer not to deviate from their desired arrival time (i.e., arrive as close as possible to their desired arrival time) while they

TABLE 1 – *List of notations*

$\mathcal{T}$	Time horizon.
i	Index of trips, $i \in N$.
x	Vector of trip lengths.
t_a	Vector of desired arrival times.
$K(x,t)$	Number of agents at time t with remaining trip distance greater than x.
$H(t)$	$:= K(0,t)$. Number of agents at time t in the network.
$v(t)$	$= V(H(t)$. Common velocity of agents at time t.
$z(t)$	Characteristic travel distance.
$T(t_d^i, x^i, t)$	Travel time of a trip started at t_d^i with trip length x^i at time t.
$m(t_a, x)$	distributions of demand with trip length x and desired arrival time t_a.
$h(x)$	distributions of initial agents with trip length x.
$f(t_a, x, t)$	distributions of departure times t with desired arrival time t_a and trip length x. *This is the unknown of the problem.*

do not spend too much time on the traffic. Note that the dependency of the cost function on the trip lengths are not emphasized in the notation, while it holds implicitly.

Let us now complete the description of the bathtub model. The data is given by the distribution of travel demand $m(t_a, x)dt_a dx$ wrt desired arrival time $t_a \in \mathcal{X}_a$ and trip length $x \in \mathcal{X}$. The unknown in the SO problem is the distributions of departure times $t \in \mathcal{T}$ with desired arrival time $t_a \in \mathcal{T}_a$ and trip length $x \in \mathcal{X}$. The resulting distribution of traveller demand is denoted as $f(t_a, x, t)dt_a dx dt$. Thus f satisfies to the following convex set of constraints $(\mathcal{K})$:

$$(\mathcal{K}) \quad \left| \begin{array}{l} \int_{\mathcal{T}} f(t_a, x, t)dt = m(t_a, x) \\ f(t_a, x, t) \geq 0 \end{array} \right. \tag{4}$$

The dynamics of the bathtub system result from the following processes: i) travellers are conserved, ii) travellers travel at speed $v_t = V(H(t))$, iii) travellers exit the system when they have travelled the trip length x (thus yielding the outflow of the system), iv) the travel demand $f(t_a, x, t)$ yields the inflow into the system. The distribution of initial agents with trip length x provides the initial condition of the system. $z(t)$ and $H(t)$ constitute the main dynamic variables. The following set of equations describes the dynamics of the system:

$$\left| \begin{array}{ll} z(t) := \int_0^t dt V(H(t)) & (5.1) \\ H(t) = h(z(t)) + \int_0^t ds\, \bar{F}(z(t) - z(s), s) & (5.2) \\ \bar{F}(x, t) = \int_{\mathcal{T}_a} dt_a\, f(t_a, x, t) & (5.3) \end{array} \right. \tag{5}$$

It can be shown (refer to Ameli *et al.* (2021a)) that (5) admits a unique solution in z and H which depends continuously on the data and initial conditions, given some regularity conditions on the data m.

The objective of the SO problem is to optimize the total travel cost of travellers. Thus the objective, denoted as $\mathcal{J}$, can be viewed as the sum over all travellers (i) of J_i given by (3), and the J_is must be calculated using (5). Thus $\mathcal{J}$ is given by

$$\begin{array}{l} \mathcal{J} \stackrel{def}{=} \int_{\mathcal{T}_a \times \mathcal{X} \times \mathcal{T}} dt_a\, dx\, dt\, f(t_a, x, t) J(t_a, x, t) \\ \left| \begin{array}{ll} J(t_a, x, t) = \alpha T(t, x) + \beta\big(t_a - t - T(t, x)\big)_+ + \gamma\big(t + T(t, x) - t_a\big)_+ & (6.1) \\ T(t, x) = z^{-1}(x + z(t) - t) & (6.2) \\ z(.) \text{ solution of (5)} & (6.3) \end{array} \right. \end{array} \tag{6}$$

Actually J is a function of t_a, x, t through z which itself is a function of f, h through (5). Thus we can also denote J as $J(f, h)$. It can be shown that J is weakly continuous wrt f, choosing as a functional setting for f either the space of bounded positive finite measures, or the space of square integrable functions $L^2(\mathcal{T}_a \times \mathcal{X} \times \mathcal{T})$. For applications and numerical approximations we shall opt for the latter functional space. Note also that in the definition of J given in (6) we could substitute the block $\beta\big(t_a - t - T(t, x)\big)_+ + \gamma\big(t + T(t, x) - t_a\big)_+$ with any other suitable convex function $\mathcal{L}$. Thus the SO problem can be stated as:

$$\min_{f \in \mathcal{K}} \mathcal{J} = \int_{\mathcal{T}_a \times \mathcal{X} \times \mathcal{T}} dt_a\, dx\, dt\, f\, J(f, h) \tag{7}$$

with $J(f, h)$ being calculated from (5) by (6.2) ,(6.3). By a Weierstrass-type argument, it is shown that (7) admits a solution (not necessarily unique) in any of the two functional spaces mentioned above. A solution in the measure space should have a better criterion value but exhibit less regularity than a solution in the L^2-space.

It can be shown that in the space $L^2(\mathcal{T}_a \times \mathcal{X} \times \mathcal{T})$, $\mathcal{J}$ admits a gradient. Further, the projector on the convex set $\mathcal{K}$ is well-defined and easy to calculate. These two facts open the way for finding numerical solutions of (7). Several discretization methods are available, based either on a particle discretization or on a cell discretization of (5), which is then used for the numerical approximation of (7).

3 PRELIMINARY RESULTS AND DISCUSSION

The numerical example illustrated by Figure 1 was calculated based on a cell discretization (discrete values of t_a, cells for x and t values). The desired arrival time takes 7 discrete values (starting 7.30 am, separated by half-an-hour). For this relatively small-sized example, convergence is achieved after some 25 iterations and finds the optimal. Convergence depends on network characteristics level of congestion.

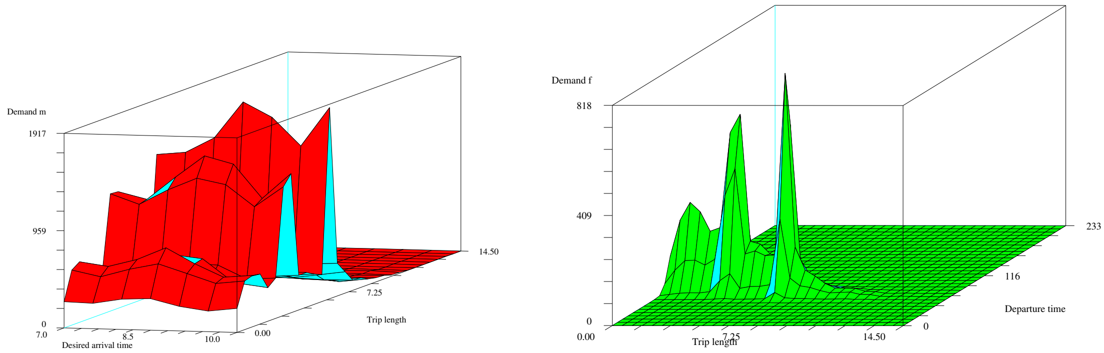

FIGURE 1 – *Example of calculation based on continuous approximation of the Lyon data (Ameli* et al.*, 2021b). The resolution is imposed by the original data. On the left: the demand m. On the right: the density of departure times f for a single arrival time (9 am).*

The SO distribution of the departure time for the trips with the desired arrival time of 9 am is shown in Figure 1. The results show that the solution for the SO does not follow any sorting pattern, e.g., FIFO and LIFO. In order to investigate further the sorting pattern and the solution characteristic, we are currently running simulations on a real scenario (Ameli *et al.*, 2021b) for the large-scale network of Lyon North (representing the morning peak hours with 62,450 trips) with trip-based dynamic implementation, and the results are consistence with the continuous test case.

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